



Beijing-Dublin International College



MIDTERM - SPRING TRIMESTER EXAMINATION - 2025/2026

MATH1002J Introduction to Analysis

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Time Allowed: 80 minutes

Instructions for Candidates

Full marks will be awarded for complete answers and correct steps to all 3 questions. The number of marks per question is indicated.

Show and explain carefully all the steps of your workings in the Answer Sheet.

If you finish the exam early, you still are required to stay until the end.

The answer sheets should provide enough space for each question. The last two pages give extra space, for all calculations, if needed.

IMPORTANT: at the end of the exam stop writing. The invigilators will allow to orderly retrieve your phones in a organized manner. You shall take a picture of your own answer sheet, and afterwards you will submit the original answer sheet to the invigilators. You will need your own solutions for the Self-Assessment Report.

Instructions for Invigilators

This examination is closed-book and closed-notes. Non-programmable calculators are permitted. No rough-work paper is to be provided to the candidate.

Students who finish the exam cannot leave before the end of the allowed time.

At the end of the exams, the students should stop writing; afterwards, they will orderly allowed to retrieve their phones and they will be given 2 minutes to quickly take pictures of their own solutions, and then submit the answer sheet.

About the Grading Scheme

The grading scheme below is given step by step. It is fairly complete. However it does not account for full details and minutiae, or alternative solutions that may exist. For these reasons:

1. Follow the solution in dark blue to check the necessary step to solve each problem.
2. Follow the “[+5, explanation...]” to decide how many marks attribute to yourself for each step/portion of the exercise.
3. If your solution was different, but still fundamentally correct, you can assign to that step/portion marks, in increasing way according to the fact that your solution is incomplete, partially complete, or fully complete.
4. If some details are missing in your solution are missing, decide the penalty according to the importance of the missing step (if not explicitly stated already in the grading scheme).
5. Check that the total mark you give to each exercise does not exceed the maximum that you can obtain for that component.
6. (Suggestion) Quickly review your self-grading after you complete the Self-Assessment Report.

1. Decide if the following statements are True or False, and justify your answer.
 - (a) Select 6 different numbers in the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then in your list there must be a pair of numbers which sums to 10. [8]
 - (b) The rule $f(x) = y$ such that $y^2 = x$ defines a function $f : \mathbb{R} \rightarrow \mathbb{R}$. [8]
 - (c) The rule $f(x) = y$ such that $y^3 = x$ defines a function $f : \mathbb{R} \rightarrow \mathbb{R}$. [8]
 - (d) The contrapositive statement of “If a function f is differentiable at c , then it is continuous at c ” (which is true from calculus) is “If a function is continuous at c , then it is differentiable at c ”. [8]
 - (e) An ant is moving on a 5×4 grid, and goes from the lower left corner to the upper right corner. If it only moves going up or right, it can reach the upper right corner of the grid in 150 ways (hint: use the binomial theorem...). [8]

(Bonus) A stock worth ¥100 can only go up by 10% or down by 10% each quarter (every 3 months). For a one year investment, we can attain the final value $100 \times 1.1^2 \times 0.9^2$ in 6 possible ways. [+4]

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SOLUTION:

[The exercise has to be evaluated as follows:

- +4 for the correct choice “True/False”
- +4 for the fully correct justification/proof (or counterexample). If no justification is given, this part is 0. If there is only a minor mistake/imprecision, +3. For multiple imprecisions but overall correct details, +2. For major parts missing or quick and wrong argument (but honest attempt), +1.

If the argument you present is different from the one(s) given in this solution, but FULLY correct, you can get full marks.

For the bonus, it is the same, but +2 for correct answer, +2 for correct justification.

There are 5 questions, plus the bonus, for a total of $5 \times 8 + 4 = 40 + 4$ marks.]

(a) True. Indeed, we can divide the set as follows:

$$P := \{\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5\}\}$$

This partition contains 5 sets, but if we select 6 elements from the original set, by the pigeonhole principle 2 must fall in the same set (not $\{5\}$!) of the partition. All such sets contain pairs of numbers which sums up to 10. $\square$

Note: we used the 5 subsets in the partition as pigeonholes and the 6 selected numbers as pigeons.

(b) False. In fact, the function is not well defined as it does not associate to each value of x a unique value $f(x)$. Indeed, $f(4)$ should be y such that $y^2 = 4$, but we have two possible answers, ± 2 . Therefore the rule does not define a function. $\square$

(c) True. Since $y \rightarrow y^3$ is injective, there is a unique solution to the equation $y^3 = x$ for all $x \in \mathbb{R}$. Therefore the rule defines a unique output, which constitutes a function from $\mathbb{R} \rightarrow \mathbb{R}$, and it is the function $\sqrt[3]{x}$. $\square$

(d) False. Indeed if we call

- P : “ f is differentiable at c ”, and
- Q : “ f is continuous at c ”,

then the original statement is $P \implies Q$, and the contrapositive statement should be $\sim Q \implies \sim P$, meaning: “If f is NOT continuous at c , then it is NOT differentiable at c ”. $\square$

- (e) False. Indeed, the and can move all the way up 5 times and right 4 times, and that would be $uuuuurrrr$. All possible ways are obtained by all permutations of that letter, which all lead to the same term u^5r^4 in the expansion of $(u + r)^9$, where $9 = 5 + 4$ total moves. The term has coefficient $\binom{9}{5} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 3 \times 7 \times 6 = 126$ ways. $\square$
- (f) **(Bonus):** True. Indeed, the order in which we multiply by 1.1 or 0.9 does not matter as long as the powers are correct. This is equivalent as finding the coefficient for $1.1^2 0.9^2$ in the expansion of $(1.1 + 0.9)^4$, which is equal to $\binom{4}{2} = 6$. $\square$
- Alternatively (similar), this is like choosing if the investment is going up “u” or down “d”, and asking that it goes up twice and down twice. Since the order is not important, it is the same as counting the different permutations of the word $uudd$ (same as the u^2d^2 coefficient of $(u + d)^4$), which are $\binom{4}{2} = 6$. $\square$
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2. The goal of this question is to fully prove the following proposition:

$\mathcal{P}$: “ $\sqrt{15}$ is an irrational number.”

- (a) Show that 15 divides the integer n^2 if and only if 15 divides the integer n (hint: be careful, 15 is not a prime number...) [15]
- (b) Explain in detail why we can write any positive rational number $q \in \mathbb{Q}$ as $q = \frac{m}{n}$ where $m, n \in \mathbb{N}$, with $\gcd(m, n) = 1$. [5]
- (c) Write the opposite statement of $\mathcal{P}$ both in words and using quantifiers. [5]
- (d) Using the results of the previous points, write a complete proof by contradiction that $\sqrt{15}$ is an irrational number. [15]

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SOLUTION:

(a) For part (a) we have to show two things:

(1) If $15 \mid n$, then $15 \mid n^2$.

This is clear, since if $n = 15k$, then $n^2 = 15^2k^2 = 15(15k^2)$, so that n^2 is divisible by 15. ([+5] for this part correctly done)

(2) If $15 \mid n^2$, then $15 \mid n$.

Assume that $15 \nmid n$, then either $3 \nmid n$ or $5 \nmid n$ (or both).

CASE 1: assume that $3 \nmid n$. Then $n = 3k \pm 1$. Therefore $n^2 = 3(k^2 \pm 2k) + 1$, i.e. $n^2 = 3\alpha + 1$ is not divisible by 3. By contrapositive, this

proves the statement in the case $3 \nmid n$.

CASE 2: assume that $5 \nmid n$. Then $n = 5k \pm 1$ or $n = 5k \pm 2$, so that $n^2 = 5(5k^2 \pm 2k) + 1$ or $n^2 = 5(5k^2 \pm 4k + 1) - 1$, i.e. $n^2 = 5k \pm 1$ which is not divisible by 5. By contrapositive, this proves the statement in the case $5 \nmid n$.

Finally, notice that when both 3 and 5 do not divide n , we can use either CASE 1 or CASE 2 above to find the conclusion. $\square$ **[+10] for the above proof (or similar) done fully correctly! Grant partial marks according to the level of details and correctness of the proof, with 0 meaning NO proof. For any attempt at the proof, although mostly incorrect, you still get +1.**

[+4] are obtained if there is a distinction in the two cases. [+3] for a correct proof in each case.

[Alternative Solution]

- (1) Thanks to the Fundamental Theorem of Arithmetic, every integer n can be written as a product of primes, in particular there is a list of prime numbers $p_1, p_2, \dots, p_r$ such that:

$$n = p_1 \cdot p_2 \cdots p_r.$$

Notice that prime numbers in the list may repeat, and we are not saying anything about the order of the (size of the) primes. Notice that:

$$n^2 = p_1^2 \cdot p_2^2 \cdots p_r^2,$$

so that if $15 = 3 \cdot 5 \mid n^2$, it must be that $3 \mid n^2$ and $5 \mid n^2$ (since $\gcd(3, 5) = 1$, or, also, they are different primes). This means that both 3 and 5 must both appear in the factorization of n^2 . Up to rearranging the primes, this must mean that $p_1 = 3$ and $p_2 = 5$, which implies that:

$$n = 3 \cdot 5 \cdot p_3 \cdots p_r = 15 \cdot p_3 \cdots p_r,$$

so that $15 \mid n$. **[10] for part (1). Noticing that 3 and 5 must both appear in the factorization of n is worth 4 marks. Noticing that $\gcd(3, 5) = 1$ or $3 \neq 5$ as a necessity for the step is worth 2 marks (marks can be attained also if it is evident that you are using this property... if it is not evident, you only get 1 out of 2 for this portion). Correctly quoting and using the Fundamental Theorem of Arithmetic from Chapter 1 is worth 4 marks. If you only use the theorem or its implications without properly quoting it, you only get 2 marks.**

(2) If $15 \mid n$ then $15 \mid n^2$. Indeed, $15 \mid n$ means that $n = 15k$ for some integer k . Therefore, $n^2 = 15(15k^2)$, so that $15 \mid n^2$. [+5] for this part correctly done.

(Other alternative proofs are to be evaluated to give a total of 15 marks, and marks need to be granted according to the level of accuracy and precision of the proof.

If you explicitly notice a bug in the original question which said: "... 15 divides the integer n^2 if and only if 15 divides n ...", and you remark that n may be not an integer from the original formulation, for this time only (next time you should ask the invigilators or me in the classroom as I recommended... i you can grant yourself up to +5 marks for your observation; whatever you get for this part, cut at 15.)

(b) Any rational number can be written as a fraction, meaning $q = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $b \neq 0$. Since we focus on positive numbers, we can ask for $a, b > 0$. Finally, if $d = \gcd(a, b)$, then $a = dm$ and $b = dn$, where $\gcd(m, n) = 1$ and $q = \frac{a}{b} = \frac{dm}{dn} = \frac{m}{n}$, so that we can simplify common factors ([+5] in total. +2 for using the definition of rational numbers as fractions correctly, +2 for noticing that $m, n > 0$, +2 for reducing correctly to $\gcd = 1$. Cut at 5

(c) We can see that:

$\mathcal{P}^{\text{opp}}$: " $\sqrt{15}$ is NOT an irrational number",

but since a number can only be rational or irrational, and it is either rational or irrational, NOT irrational is equivalent to rational, so that we can rewrite it and state the opposite statement as:

$\mathcal{P}^{\text{opp}}$: " $\sqrt{15}$ is a rational number".

Since every rational number can be written as a fraction, we can rewrite the opposite statement as follows

$\mathcal{P}^{\text{opp}}$: " $\sqrt{15} = \frac{m}{n}$, where $m, n \in \mathbb{N}$."

([+5] for this. The $\gcd = 1$ condition is not necessary to create the opposite statement. If it is introduced and this is the conclusion:

$\mathcal{P}^{\text{opp}}$: " $\sqrt{15} = \frac{m}{n}$, where $m, n \in \mathbb{N}$, and $\gcd(m, n) = 1$."

the full marks are still given.

(d) For sake of contradiction, we assume that $\sqrt{15} = \frac{m}{n}$ with $m, n \in \mathbb{N}$, and $\gcd(m, n) = 1$.

([+4] for the correctly stated assumption in order to start the contradiction. Remove

marks if details are missing (e.g. $\gcd(m, n) = 1$ missing is -1, not stating that we are doing an argument by contradiction is -2...))

Then $m^2 = 15n^2$, which means that $15 \mid m^2$, so that $15 \mid m$ by part (a). Therefore we can write $m = 15a$.

([+4]. +2 to identify that m^2 must be divisible by 15. +2 to identify that m must be divisible by 15 using part (a).

If you use divisibility by 3 or by 5 instead, i.e. 3 (or 5) divides m^2 therefore 3 (or 5) divides m , it is ok too (we have seen this for prime numbers so it is ok)) Then $15a^2 = n^2$, and since 15 divides n^2 even, then 15 divides n so that $n = 15b$ is even as well.

([+4]. +2 to identify that n^2 must be divisible by 15. +2 to identify that n must be divisible by 15 using part (a).

If you use divisibility by 3 or by 5 instead, i.e. 3 (or 5) divides m^2 therefore 3 (or 5) divides m , it is ok too (we have seen this for prime numbers so it is ok)) Therefore 15 divides both m and n , so that $15 \mid \gcd(m, n) = 1$, which is absurd (15 does not divide 1 in $\mathbb{Z}$!). Since this contradicts the initial assumption in the proof that $\sqrt{15}$ is rational, then it must be irrational. $\square$

([+4], for the correct identification of the contradiction/absurd and conclusion of the argument. The total is 16, cut at 15.) $\square$

3. Here below there are 5 statements. You can prove them in any order you want and with the technique you prefer. The first statement you fully prove is worth 8 marks, the second 6 marks, the third 4 marks, the fourth 2 marks. The last can give you a +2 bonus marks. [20+2]

(a) For every $\alpha \neq 1$, $1 + \alpha + \dots + \alpha^{n-1} = \frac{\alpha^n - 1}{\alpha - 1}$.

(b) The function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 1 + e^{-x^2}$, has image $\text{Im}(f) = (1, 2]$.

(c) The number $\sqrt{5} + \sqrt{3}$ is irrational.

(d) The rule $x \mapsto f(x) := \lim_{n \rightarrow \infty} x^n$ defines a function $f : [0, 1] \rightarrow [0, 1]$ which is not invertible, even if x^n is invertible for all n .

(e) The rule $f(x) = e^{x-1} - \ln(x)$ defines a function from $[1, +\infty)$ to $[1, +\infty)$ which is invertible.

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SOLUTION:

In order to mark this question, one way could be the following: give a mark from 0 to 24 according to the grading scheme below for each question. Then order the 5 marks in decreasing order: $\alpha \geq \beta \geq \gamma \geq \delta \geq \varepsilon$. The grade will be determined

by the formula $\frac{\alpha}{3} + \frac{\beta}{4} + \frac{\gamma}{6} + \frac{\delta}{12} + \frac{\varepsilon}{12}$. For example, if the grades that you give are:

$$(a) 12, \quad (b) 18, \quad (c) 22, \quad (d) 6, \quad (e) 14,$$

then:

$$(\alpha, \beta, \gamma, \delta, \varepsilon) = (22, 18, 12, 14, 6) \implies \frac{22}{3} + \frac{18}{4} + \frac{12}{6} + \frac{14}{12} + \frac{6}{12} = 15.5$$

Therefore, the grading below goes up to 120, but the final grade is to be decided using the above algorithm and it is at most $\frac{24}{3} + \frac{24}{4} + \frac{24}{6} + \frac{24}{12} + \frac{24}{12} = 20 + 2$.

(a) We proceed by induction ([+2] for a correct choice of method). Indeed, for the induction base we prove the case $n = 1$: we easily see that $1 = \frac{\alpha-1}{\alpha-1}$, which proves the base ([+6] for the induction base).

Let us now assume that, by induction hypothesis, we have for some $k > 1$ that:

$$1 + \alpha + \dots + \alpha^{k-1} = \frac{\alpha^k - 1}{\alpha - 1}.$$

([+6] for the induction hypothesis correctly stated) Therefore to prove the statement for $n = k + 1$ we can see that:

$$1 + \alpha + \dots + \alpha^{k-1} + \alpha^k = \frac{\alpha^k - 1}{\alpha - 1} + \alpha^k = \frac{\alpha^k - 1 + \alpha^{k+1} - \alpha^k}{\alpha - 1} = \frac{\alpha^{k+1} - 1}{\alpha - 1},$$

which is the statement for $n = k + 1$ ([+6] for the correct proof of the step). Here in the second step we used the induction hypothesis ([+2] for clearly stating where the induction hypothesis is used. Ok both in a sentence after the formula, or also if clearly indicated inside the formula).

This (base, hypothesis, step) proves the statement by mathematical induction ([+2] for stating explicitly the correct conclusion. The total should be 24).

(b) First of all, let us notice that $e^{-x^2} > 0$, so that $f(x) = 1 + e^{-x^2} > 1$ for all x ([+4] for noticing this and clearly explaining it). Second, since $-x^2 \leq 0$ for all x , it is clear that $e^{-x^2} \leq 1$ for all x so that $f(x) = 1 + e^{-x^2} \leq 2$ ([+4] for noticing this bound, and explaining it clearly). Therefore $\text{Im}(f) \subset (1, 2]$ ([+4] for correctly summing up the two previous observations with this inclusion).

Finally, take any $y \in (1, 2]$ we can see that

$$1 + e^{-x^2} = y \in (1, 2] \iff e^{-x^2} = y - 1 \in (0, 1] \iff -x^2 = \ln(y - 1) \in (-\infty, 0] \\ \implies x = \pm \sqrt{-\ln(y - 1)}$$

which clearly has 2 solutions since $-\ln(y - 1) \geq 0$ ([+8] for this procedure and all clearly stated steps). This shows that $(1, 2] \subset \text{Im}(f)$ ([+4] for the correct conclusion of this part). Since $\text{Im}(f) \subset (1, 2] \subset \text{Im}(f)$, then $\text{Im}(f) =$

$(1, 2]$, which concludes the proof ([+2] for stating the correct conclusion from the previous steps. The total goes to 26, cut at 24. has solutions since it is equivalent to $e^{-x^2} = y - 1 \in (0, 1]$, so that $-x^2 =$

- (c) Clearly, this statement is true: if $q = \frac{m}{n}$ is rational, $q^2 = \frac{m^2}{n^2}$ is rational too (it is still a fraction) ([+6] for this statement with proof. If the proof is missing, only +4).

Therefore, if $(\sqrt{3} + \sqrt{5})^2$ is irrational, it must be true that $\sqrt{3} + \sqrt{5}$ is irrational too (by contrapositive of the previous statement) ([+6] for this contrapositive clearly stated and understood. Only +4 if it is used correctly, but not explicitly stated).

But we can see that:

$$(\sqrt{3} + \sqrt{5})^2 = 8 + 2\sqrt{15},$$

([+4] for the correct square) and $8 + 2\sqrt{15}$ is irrational IFF $2\sqrt{15}$ is irrational (since 8 is rational) ([+2] for the correct reduction to the irrationality of $2\sqrt{15}$). Since $\sqrt{15}$ is irrational by question 2 ([+2] if correctly stated and quoted), and 2 is rational, we can conclude that $2\sqrt{15}$ is irrational ([+2] for the correct reduction to the irrationality of $\sqrt{15}$), therefore $8 + 2\sqrt{15}$ is irrational too, so that its square root, $\sqrt{3} + \sqrt{5}$, must be irrational as well ([+4] for the correct chain-recap of the steps/conclusion; the total goes up to 26, cut at 24).

- (d) For each function $f_n(x) = x^n$ we can notice that:

$$f'_n(x) = nx^{n-1} > 0, \quad \forall x \in (0, 1).$$

Therefore f_n is monotonic increasing, and continuous, which means that it is injective for all n ([+4] for the correct proof of injectivity for f_n). Since $f_n(0) = 0$ and $f_n(1) = 1$ for all n , and since each $f_n(x)$ is continuous, by the intermediate value theorem (consequences seen in Chapter 3), we have that $\text{Im}(f_n) = [0, 1]$, i.e. f_n is surjective for all n ([+4] for the correct proof of surjectivity). Since f_n is injective and surjective for all n , it is bijective for all n , therefore invertible ([+4] for the correct conclusion/derivation of f_n being bijective/invertible).

It is clear that:

$$\lim_{n \rightarrow \infty} x^n = 0, \quad \forall x \in [0, 1), \quad \text{and} \quad \lim_{n \rightarrow \infty} 1^n = 1,$$

so that:

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}$$

([+6] for the correct derivation of the function $f(x)$). Clearly $f(0) = f(0.5) = 0$, so that the function f is NOT injective ([+4] for f not injective plus proof). Since f is not injective, it is not bijective, therefore it is not invertible ([+4] for the correct conclusion about f . The total goes up to 26, cut at 24).

(e) Since the function is differentiable,

$$f'(x) = e^{x-1} - \frac{1}{x}.$$

([+4] for the correct derivative) We notice that $f'(1) = 0$, but then since e^{x-1} and $-\frac{1}{x}$ are strictly increasing, then $f'(x) > 0$ ([+4] for the correct proof that the derivative is positive, even if different from this proof). Therefore $f(x)$ is strictly increasing, so that it is injective ([+4] for the correct conclusion that f is increasing/injective).

Clearly, $f(1) = e^0 - \ln(1) = 1$, so that $1 \in \text{Im}(f)$ ([+4] for noticing that 1 is in the image of f). Moreover, since $\lim_{x \rightarrow +\infty} f(x) = +\infty$, so that for all $b > 1$ there is an $M > 0$ such that for all $x > M$, $f(x) > b$. In particular, there is x_0 such that $b_0 = f(x_0) > b$. Therefore, $b \in [1, b_0] \subset \text{Im}(f)$ for all $b > 1$, which proves that f is surjective ([+4] for a statement and proof that f is surjective correctly applying the results as seen in class).

Since f is injective and surjective, it is bijective, which is equivalent to being invertible ([+4] for the correct conclusion and brief explanation. The total is 26, cut at 24).

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