



Beijing-Dublin International College



MIDTERM - SPRING TRIMESTER EXAMINATION - 2023/2024

MATH1002J Introduction to Analysis

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Time Allowed: 80 minutes

Instructions for Candidates

Full marks will be awarded for complete answers and correct steps to all 4 questions. The number of marks per question is indicated.

Show and explain carefully all the steps of your workings in the Answer Sheet.

At the end of the exam you need to submit **ONLY** the Answer Sheet (keep the questions sheet).

If you finish the exam early, you still are required to stay until the end.

IMPORTANT: at the end of the exam stop writing. The invigilators will allow to orderly retrieve your phones in an organized manner. You shall take a picture of your own answer sheet, and afterwards you will submit the original answer sheet to the invigilators. You will need your own solutions for the Self-Assessment Report.

Instructions for Invigilators

This examination is closed-book and closed-notes. Non-programmable calculators are permitted. No rough-work paper is to be provided to the candidate.

Students who finish the exam cannot leave before the end of the allowed time.

At the end of the exams, the students should stop writing; afterwards, they will orderly allowed to retrieve their phones and they will be given 2 minutes to quickly take pictures of their own solutions, and then submit the answer sheet.

1. In this question you have to find where is the **mistake** in the given “proof” of the clearly paradoxical statement: “*All points are on the same line*”.

— BEGINNING OF THE PROOF —

- Proposition $\mathcal{P}(n)$: “if we have a set of n points, then they are collinear (on the same line)”.
- Induction Base: for $n = 1$ the statement $\mathcal{P}(n)$ becomes “if there is only one point, it is on a line”, which is obviously true.
- Induction Hypothesis: assume that $\mathcal{P}(k)$ is true, i.e. that any set of k points, is on a single line.
- Induction Step: We want to use $\mathcal{P}(k)$ to show $\mathcal{P}(k + 1)$ is true. Take any set of $k + 1$ points; call this set $\{P_1, P_2, P_3, \dots, P_k, P_{k+1}\}$. Consider its two subsets $A = \{P_1, P_2, P_3, \dots, P_k\}$ and $B = \{P_2, P_3, \dots, P_k, P_{k+1}\}$. Each is a set of k points, therefore, using the induction hypothesis, we conclude that each of this two sets must be on the same line, i.e. all points in A must be on the line ℓ_A , and all points of B must be on the line ℓ_B . But the two sets overlap because their intersection is $\{P_2, \dots, P_k\}$ which then belongs to both lines ℓ_A and ℓ_B , so they must be the same, therefore there must be only one line containing all $k + 1$ points.
- Since we did all the necessary steps, by mathematical induction the statement $\mathcal{P}(n)$ is true for all $n \geq 1$, so all sets of n points must be collinear.

— END OF THE PROOF —

Carefully read the argument, decide if the result is true or false. Explain and defend your decision using clear logical arguments. [10]

2. Prove that $\sqrt{10}$ is an irrational number. [30]

3. For any number $c \in \mathbb{R} \setminus \{0\}$, the function $f_c : \mathbb{R} \setminus \{-c\} \rightarrow \mathbb{R}$ given by

$$f_c(x) = \frac{c^{2024}}{x + c} \left(\binom{2024}{0} + \binom{2024}{1} \frac{x}{c} + \dots + \binom{2024}{2} \left(\frac{x}{c}\right)^2 + \dots + \binom{2024}{2024} \left(\frac{x}{c}\right)^{2024} \right)$$

Find the image $\text{Im}(f_c)$ (notice: you do not have to choose a number c). [30]

4. Decide if the following statements are True or False, and justify your answer.

- (a) If $f : [0, 10] \rightarrow [3, 5]$ is injective and $g : [1, 6] \rightarrow [0, 1]$ is injective, then the composition $g \circ f$ and it is injective too. [6]

(b) The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by:

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 1, \\ x, & \text{if } x \geq 1, \end{cases}$$

is invertible. [6]

(c) The rule $f(a/b) = b$ gives a function $f : \mathbb{Q} \rightarrow \mathbb{Z}$? [6]

(d) If I have 7 players, I can form a 2 players Badminton team in 25 different ways. [6]

(e) The equation $y = x + \frac{8}{x^2}$ has only 3 solutions with $x, y \in \mathbb{Z}$. [6]

(Bonus) There is no injective function from $(0, 1)$ to $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \dots$ (product of infinitely many copies of $\mathbb{N}$). [+6]

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