

NOTE: Answer **ALL** questions.

Time allowed is **95** minutes.

The exam paper has **2** sections on **4** pages, with a full score of 100 marks.

You are required to use the provided **Examination Book** only for answers.

Section A: Fill-in-the-blank Questions

This section is worth a total of **75** marks.

1. (5 marks) Evaluate

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} = \underline{\hspace{2cm}}$$

2. (5 marks) Find the limit

$$\lim_{x \rightarrow \infty} \frac{\arctan x}{x} = \underline{\hspace{2cm}}$$

3. (5 marks) Find the limit

$$\lim_{x \rightarrow 0} \frac{\sqrt{1-x} - 1}{x} = \underline{\hspace{2cm}}.$$

4. (5 marks) Let $f(x)$ be the function

$$f(x) = \frac{1}{x^2 - 1}.$$

Find the higher order derivative $f^{(2022)}(x) = \underline{\hspace{2cm}}.$

5. (5 marks) Let $f(x)$ be the function

$$f(x) = \ln \left(x + \sqrt{x^2 + 1} \right).$$

Find the differential $df(x) = \underline{\hspace{2cm}}.$

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6. (5 marks) Find the limit

$$\lim_{n \rightarrow \infty} \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n} = \underline{\hspace{2cm}}.$$

7. (5 marks) Given

$$\lim_{x \rightarrow 0} \frac{\ln [1 + h(x)]}{\arctan x} = 3,$$

find the limit

$$\lim_{x \rightarrow 0} \frac{h(x)}{x} = \underline{\hspace{2cm}}.$$

8. (5 marks) Find the limit

$$\lim_{x \rightarrow \infty} \frac{(2x - 1)^{2023}}{(3 - 2x)^{2023}} = \underline{\hspace{2cm}}.$$

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9. (5 marks) The equation of the vertical asymptote to the given curve $y = \frac{x^2 - 1}{x^2 - 3x - 4}$ can be expressed as

_____.

10. (5 marks) Find the equations of the tangent line to the curve given by $x + \sqrt{xy} = 6$ at the point of contact $(4, 1)$ _____.

11. (5 marks) The following function

$$f(x) = \begin{cases} (1 - 2x)^{\frac{2}{x}} & x \neq 0 \\ A & x = 0 \end{cases}$$

is continuous at $x = 0$, then $A = \underline{\hspace{2cm}}.$

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12. (6 marks) Given

$$\begin{cases} x = 1 - t^2, \\ y = t - t^3, \end{cases} \quad t \text{ being a parameter, } t \in \mathbb{R},$$

evaluate

$$\frac{d^2y}{dx^2} = \underline{\hspace{2cm}}$$

13. (7 marks) Find $\frac{dy}{dx}$ by taking logarithm and then implicit differentiation, where

$$y = x^x \cdot \sqrt{x+1}, \quad x > 0$$

14. (7 marks) Evaluate

$$\lim_{x \rightarrow 0} \left(\frac{1}{\ln(x+1)} - \frac{1}{x} \right),$$

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Section B: Extended Answer Questions

This section is worth a total of **25** marks.

- 15.** (8 marks) Suppose $f(x)$ is a continuous function over the interval $[0, 2]$. $f(x)$ is derivable on the point $(0, 2)$, with $f(2) = 0$.

Try to prove that there exists at least one number $\xi \in (0, 2)$, such that

$$f'(\xi) = -\frac{5f(\xi)}{\xi}.$$

- 16.** (10 marks) Given $f(x) = x^3 - 3x^2 - 9x + 5$, , find its monotonic intervals, local maximum values and minimum values, its interval of concave up and concave down, and its inflection point.

- 17.** (7 marks)Prove that the equation $x^5 + x - 3 = 0$ has only one positive root.