NOTE: Answer **ALL** questions.

Time allowed is **95** minutes.

The exam paper has **2** sections on **4** pages, with a full score of 100 marks.

You are required to use the provided **Examination Book** only for answers.

Section A: Fill-in-the-blank Questions

This section is worth a total of **75** marks.

1. (5 marks) Evaluate

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2}=\underline{\qquad\qquad}$$

$\lim_{x\to 0}\frac{e^x-1}{2x}=\frac{e^x}{2}=\frac{1}{2}$

2. (5 marks) Find the limit

$$\lim_{x\to\infty}\frac{\arctan x}{x}=\underline{\qquad\qquad}$$

3. (5 marks) Find the limit

$$\lim_{x\to 0}\frac{\sqrt{1-x}-1}{x}=\underline{\qquad\qquad}.$$

$\frac{(1-x)^{\frac{1}{2}}-1}{x}=\frac{-\frac{1}{2}x}{x}=-\frac{1}{2}$

4. (5 marks) Let $f(x)$ be the function

$$f(x)=\frac{1}{x^2-1}.$$

Find the higher order derivative $f^{(2022)}(x)=$ _____.

$\frac{1}{(x+1)(x-1)}=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$

$\left(\frac{1}{x-1}\right)'=\frac{-1}{(x-1)^2}$

$=\frac{1}{2}\left[(-1)^n n!\frac{1}{(x-1)^{n+1}}-(-1)^n n!\frac{1}{(x+1)^{n+1}}\right]$

5. (5 marks) Let $f(x)$ be the function

$$f(x)=\ln\left(x+\sqrt{x^2+1}\right).$$

Find the differential $df(x)=$ _____.

$\frac{1}{x+\sqrt{x^2+1}}\cdot\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)dx$

$=\frac{1}{x+\sqrt{x^2+1}}\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}dx=$

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6. (5 marks) Find the limit

$$\lim_{n\to\infty}\frac{2^{n+1}+3^{n+1}}{2^n+3^n}=\underline{\qquad 3 \qquad}.$$

7. (5 marks) Given

$$\lim_{x\to 0}\frac{\ln\left[1+h(x)\right]}{\arctan x}=3,$$

find the limit

$$\lim_{x\to 0}\frac{h\left(x\right)}{x}=\underline{\qquad 3 \qquad}.$$

8. (5 marks) Find the limit

$$\lim_{x\to\infty}\frac{(2x-1)^{2023}}{(3-2x)^{2023}}=\underline{\qquad -1 \qquad}.$$

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9. (5 marks) The equation of the vertical asymptote to the given curve $y=\dfrac{x^2-1}{x^2-3x-4}$ can be expressed as

$\underline{\qquad x=4 \qquad}$.

10. (5 marks) Find the equations of the tangent line to the curve given by $x+\sqrt{xy}=6$ at the point of contact $(4,1)$ $\underline{\qquad\qquad}$.

11. (5 marks) The following function

$$f(x)=\begin{cases}(1-2x)^{\frac{2}{x}} & x\neq 0\\ A & x=0\end{cases}$$

is continuous at $x=0$, then $A=\underline{\qquad e^{-4} \qquad}$.

12. (6 marks) Given

$$\begin{cases} x = 1 - t^2, \\ y = t - t^3, \end{cases} \quad t \text{ being a parameter, } t \in \mathbb{R},$$

evaluate

$$\frac{d^2y}{dx^2} = _____$$

$\frac{dx}{dt} = -2t$

$\frac{dy}{dt} = 1 - 3t^2$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{-2t} =$

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy/dx}{dx/dt} = \frac{-6t(-2t) - 2(1-3t^2)}{4t^2}$

$= \frac{12t^2 + 2 - 6t^2}{-8t^3}$

$= \frac{6t^2+2}{-8t^3}$

$= \frac{3t^2+1}{-4t^3}$

13. (7 marks) Find $\frac{dy}{dx}$ by taking logarithm and then implicit differentiation, where

$$y = x^x \cdot \sqrt{x+1}, \; x > 0$$

$\ln y = x \ln x + \frac{1}{2}\ln(x+1)$

$\frac{1}{y} \cdot y' = \ln x + x \cdot \frac{1}{x} + \frac{1}{2}\frac{1}{x+1}$

$y' = \left[x^x \sqrt{x+1}\right]\left[\ln x + 1 + \frac{1}{2x+2}\right]$

14. (7 marks) Evaluate

$$\lim_{x \to 0}\left(\frac{1}{\ln(x+1)} - \frac{1}{x}\right),$$

$\lim_{x\to 0} \frac{x - \ln(x+1)}{x\ln(x+1)}$

$= \lim_{x \to 0} \frac{1 - \frac{1}{x+1}}{2x} = \frac{\frac{x}{x+1}}{2x}$

$= \frac{1}{2(x+1)} = \frac{1}{2}$

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Section B: Extended Answer Questions

This section is worth a total of **25** marks.

15. (8 marks) Suppose $f(x)$ is a continuous function over the interval $[0, 2]$. $f(x)$ is derivable on the point $(0, 2)$, with $f(2) = 0$.

Try to prove that there exists at least one number $\xi \in (0, 2)$, such that

$$f'(\xi) = -\frac{5f(\xi)}{\xi}.$$

Let $F(x) = x^5 f(x)$

$F(0) = 0 \quad F(2) = 0$

according to Rolle's Theorem.

$\therefore \exists \xi \in (0, 2) \quad F'(\xi) = 0$

$\xi f'(\xi) + 5f(\xi) = 0$

$5\xi^4 f(\xi) + \xi^5 f'(\xi) = 0$

$5f(\xi) + \xi f'(\xi) = 0$

$f'(\xi) = -\frac{5f(\xi)}{\xi}$

$27 - 27 - 27 + 5$

16. (10 marks) Given $f(x) = x^3 - 3x^2 - 9x + 5$, , find its monotonic intervals, local maximum values and minimum values, its interval of concave up and concave down, and its inflection point.

$f'(x) = 3x^2 - 6x - 9$

$= 3(x^2 - 2x - 3)$

$= 3(x - 3)(x + 1)$

$f'(x) = 0 \quad x = 3 \text{ or } 1$

$f''(x) = 3(2x - 2)$

local max : $f(1) = -6$ 10

local min : $f(3) = -22$

increase : $(-\infty, 1) \cup (3, +\infty)$

decrease : $(1, 3)$

inflection point $(1, -6)$

CU : $(-\infty, 1)$ CD : $(1, +\infty)$

17. (7 marks)Prove that the equation $x^5 + x - 3 = 0$ has only one positive root.

$f(x) = x^5 + x - 3$

$f'(x) = 5x^4 + 1$

$6 \times 2 = 32 + 2$

$f'(x) > 0$ when $x \in (-\infty, +\infty)$

$\therefore f(x)$ increase in $(-\infty, +\infty)$

$f(0) = -3 \quad f(2) = 31$

$f(0) \cdot f(2) < 1$ according to Zero point Theorem

$f(x)$ only have one zero point

$\therefore x^5 + x - 3 = 0$ only one positive root.