Beijing-Dublin International College

SEMESTER II FINAL EXAMINATION – 2023/2024

Beijing-Dublin International College
BDIC1042J Maths 3 (Advanced Mathematics; Finance)

PRINCIPAL OF COLLEGE: Wenying WU
MODULE LECTURER: Xin LIU

Time Allowed: 95 minutes

Instructions for Candidates

Your answers to ALL the questions should be written on the Answer Book.

BJUT Student ID: _____ **UCD Student ID: _____**

I have read and clearly understand the Examination Rules of both Beijing University of Technology and University College Dublin. I am aware of the Punishment for Violating the Rules of Beijing University of Technology and/or University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I accept the punishment thereof.

Honesty Pledge：_____ (Signature)

Instructions for Invigilators

No calculators are permitted. NO dictionaries are permitted.
No rough-work paper is to be provided for candidates.

NOTE: The exam paper has **4** sections on **8** pages, with a full score of **100** marks.

SECTION A — TRUE-OR-FALSE QUESTIONS

In each question, judge the statement is **True** or **False**.
Write your answers on the **Answer Book** provided.
This section is worth a total of **6** marks, with each question worth **2** marks.

1. Let S be a surface in three dimensions. Then at every point P on S, there exists a unique direction $\boldsymbol{n}$ normal to S, and thus a unique tangent plane of S at P.

True **False**

2. Let S be a surface in three dimensions, and P a point on S. If the normal direction $\boldsymbol{n}$ of S at P is unique, then the tangent plane of S at P is unique.

True **False**

3. Let Σ be a plane in three dimensions, and P a point on Σ. Then we can find only one surface S that has Σ as its tangent plane at P.

True **False**

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SECTION B — MULTIPLE CHOICE QUESTIONS

In each question, choose **at most one** option.
Write your answers on the **Answer Book** provided.
This section is worth a total of **15** marks, with each question worth **3** marks.

4. How to obtain the shape of the following paraboloid in space?

$$z = 4x^2 + y^2$$

(a) Step 1: Starting from a planar curve $z = x^2$.
Step 2: Rotating about the z-axis to obtain: $z = x^2 + y^2$.
Step 3: Doing compression in the x-direction by $x \to 2x$, to obtain: $z = (2x)^2 + y^2$.

(b) Step 1: Starting from a planar curve $z = x^2$.
Step 2: Rotating about the x-axis to obtain: $z = x^2 + y^2$.
Step 3: Doing compression in the x-direction by $x \to 2x$, to obtain: $z = (2x)^2 + y^2$.

(c) Step 1: Starting from $z = 4x^2 - y^2$.
Step 2: Doing reflection by $y \to -y$, to obtain: $z = 4x^2 + y^2$.

(d) Step 1: Starting from a plane $z = 2x + y$.
Step 2: Rotating about the line $x + y + z = 0$ to obtain: $z = (2x)^2 + y^2$.

5. On a two dimensional plane endowed with the Cartesian coordinates (x, y), let $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ be the partial derivative operators with respect to x and y, respectively.

Which of the following computations is correct?

(a) $\frac{\partial}{\partial x}\left(\frac{y}{x}\frac{\partial}{\partial y}\right) = \frac{y}{x}\frac{\partial^2}{\partial x \partial y}$.

(b) $\frac{\partial}{\partial x}\left(\frac{y}{x}\frac{\partial}{\partial y}\right) = \frac{y}{x}\frac{\partial^2}{\partial y \partial x} - \frac{y}{x^2}\frac{\partial}{\partial y}$.

(c) $\frac{\partial}{\partial y}\left(\frac{y}{x}\frac{\partial}{\partial x}\right) = \frac{1}{x}\frac{\partial^2}{\partial y \partial x}$.

(d) $\frac{\partial}{\partial y}\left(\frac{y}{x}\frac{\partial}{\partial x}\right) = \frac{y}{x}\frac{\partial^2}{\partial y \partial x} - \frac{y}{x^2}\frac{\partial}{\partial y}$.

6. In terms of the two-dimensional Cartesian coordinate system, (x, y), the Laplacian operator is given by $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Introducing the polar coordinate system (ρ, θ) through the transformation

$$\begin{cases} x &= \rho\cos\theta, \\ y &= \rho\sin\theta, \end{cases} \qquad \text{i.e.,} \qquad \begin{cases} \rho &= \sqrt{x^2+y^2}, \\ \theta &= \arctan\frac{y}{x}, \end{cases}$$

one can express the Laplacian operator Δ in terms of the ρ and θ coordinates as _____.

(a) $\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}$.
(b) $\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial\theta^2}$.
(c) $\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial}{\partial\rho} + \frac{1}{\rho}\frac{\partial^2}{\partial\theta^2}$.
(d) $\Delta = \frac{1}{\theta}\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial^2}{\partial\theta^2}$.

7. The following quadratic surface is a _____:

$$x^2 + 2z^2 - 6x - y + 10 = 0.$$

(a) sphere; (b) ellipsoid; (c) hyperboloid; (d) paraboloid.

8. The total differential of a two-variable function $z = \left(x^2 - y^2\right)e^y$ is given by

(a) $dx + dy = \left(x^2 - y^2\right)e^y dz$;
(b) $dz = \left(x^2 - y^2 + 2x - 2y\right)\left[e^y dx + \left(x^2 - y^2\right)dy\right]$;
(c) $dz = e^y\left[2xdx + \left(x^2 - y^2 - 2y\right)dy\right]$;
(d) $dz = 2xe^y dx + \left(x^2 - y^2 - 2y\right)dy$.

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SECTION C — GAP-FILLING QUESTIONS

Write your answers on the **Answer Book** provided.
This section is worth a total of **24** marks, with each question worth **4** marks.

9. Let C be a curve in space

$$C : \begin{cases} x + y^2 - z^2 &= 0, \\ x + y - 2z &= 0. \end{cases}$$

The equation of the projection of C onto the xz-plane is _____.

10. On a hot day, extreme humidity makes us think the temperature is higher than it really is. Hence a quantity, *Heat Index*, denoted as I, is devised to describe the combined effects of temperature and humidity, i.e., $I = f(T, H)$. The following table gives some data of I depending on T and H.

Heat Index (I) as a function of temperature (T) and humidity (H)
(T = Temperature, unit as °C; H = Humidity, unit as percent)

T \ H	50	55	60	65	70	75	80	85	90
32	96	98	100	103	106	109	112	115	119
33	100	103	105	108	112	115	119	123	128
34	104	107	111	114	118	122	127	132	137
35	109	113	116	121	125	129	135	141	146
36	114	118	123	127	132	138	144	150	157
37	119	124	129	135	141	147	154	161	168

At the point $T = 35°\text{C}$ and $H = 70$ percent, use the data from the table to approximate the following partial derivatives:

$$\frac{\partial I}{\partial T} \approx \frac{\Delta I}{\Delta T} = _____, \qquad \frac{\partial I}{\partial H} \approx \frac{\Delta I}{\Delta H} = _____,$$

where ΔT and ΔH are the minimum variance of T and H, respectively, near the studied point.

11. Evaluate the limit:

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{3}xy}{x+\sqrt{3}y} = _____.$$

12. Evaluate the limit:

$$\lim_{(x,y)\to(0,0)} x^2y^2 \ln\left(x^2+y^2\right) = _____.$$

13. Let $f = axy$, where a is a constant. y is a function depending on x through the relation

$$x^2 + a^2 = y^2.$$

Evaluate $\frac{df}{dx} = _____$.

14. Consider a function $z = f(x, y)$ and the following three vectors in the xy-plane,

$$\boldsymbol{u}_1 = \boldsymbol{i} + 2\boldsymbol{j}, \qquad \boldsymbol{u}_2 = 2\boldsymbol{i} + 2\boldsymbol{j}, \qquad \boldsymbol{u}_3 = 3\boldsymbol{i} - 4\boldsymbol{j}.$$

If the directional derivatives in $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are $D_{\hat{\boldsymbol{u}}_1} f = -\frac{1}{\sqrt{5}}$ and $D_{\hat{\boldsymbol{u}}_2} f = 0$, respectively, try to compute the directional derivative along $\boldsymbol{u}_3$: $D_{\hat{\boldsymbol{u}}_3} f = _____$.

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SECTION D — EXTENDED ANSWER QUESTIONS

Write your answers on the **Answer Book** provided.
This section is worth a total of **55** marks. The marks of each question are as shown.

15. (9 marks) Consider a hill in a shape described by the function

$$z = f(x, y) = x^2 y^3 - 4y.$$

Let $P(2, -1, 0)$ be a point on the hill.

(a) Find the fastest increasing direction at P, denoted as $\hat{\boldsymbol{l}}_1$, and the directional derivative in that direction, denoted as $D_{\hat{\boldsymbol{l}}_1} f$.

(b) Find the fastest decreasing direction at P, denoted as $\hat{\boldsymbol{l}}_2$, and the directional derivative in that direction, denoted as $D_{\hat{\boldsymbol{l}}_2} f$.

(c) Find the direction $\hat{\boldsymbol{l}}_3$ at P along which the directional derivative has zero change.

16. (8 marks) Consider a surface in space

$$S: \quad F(x, y, z) = 0.$$

Below $s, t \in \mathbb{R}$ serve as string parameters.

(a) What is the condition for a curve $C: \begin{cases} x = \phi(t), \\ y = \psi(t), \\ z = \omega(t) \end{cases}$ if C is a curve residing within S?

(b) Furthermore, let $\tilde{C} \begin{cases} \tilde{x} = \tilde{\phi}(s), \\ \tilde{y} = \tilde{\psi}(s), \\ \tilde{z} = \tilde{\omega}(s) \end{cases}$ be another curve residing within the surface S.

Suppose C and $\tilde{C}$ are different curves, and they intersect at a point $P_0(x_0, y_0, z_0)$.

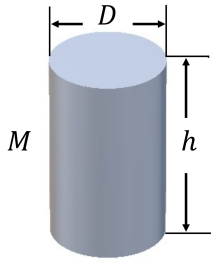
Let $\boldsymbol{v}_0$ and $\tilde{\boldsymbol{v}}_0$ be the tangent directions of C and $\tilde{C}$ at P_0, respectively, which are not parallel to each other, i.e., $\boldsymbol{v}_0 \nparallel \tilde{\boldsymbol{v}}_0$. Let $\boldsymbol{n}_0$ be the normal direction to S at P_0.

Try to prove: $\boldsymbol{n}_0$ is parallel (or anti-parallel) to $\boldsymbol{v}_0 \times \tilde{\boldsymbol{v}}_0$.

*Hint: $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c}$.

17. (8 marks) (*Error in physical experiments*)

Consider an experiment to measure the density of a type of metal. A sample is as follows, with the shape being a regular cylinder. The transection is a standard disk, with diameter measured to be D. The height of the cylinder is measured to be h. The mass of the sample is scaled to be M.



With these data, the density of this metal is obtained as

$$\rho = \rho(M, D, h) = \frac{M}{\frac{\pi}{4} D^2 h}.$$

(a) Regarding M, D and h as the variables, find the total differential of ρ.

(b) Suppose the errors of M, D and h are: (below "*unit*" means an appropriate unit)

$$\Delta M = \pm 0.05 \text{ unit}, \qquad \Delta D = \pm 0.002 \text{ unit}, \qquad \Delta h = \pm 0.002 \text{ unit},$$

and your computation gives the error of measurement for the density ρ:

$$\Delta\rho = 5.00\Delta M - 10.000\Delta D + 80.000\Delta h.$$

Try to **evaluate/compute explicitly the error $\Delta\rho$.**

18. (9 marks) Classify every critical point of a function

$$z = f(x, y) = x^4 + y^4 - 4xy + 1, \qquad x, y \in \mathbb{R}$$

to be a local maximum/minimum point or a saddle point.

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19. (8 marks) [Business — Maximizing production] A computer company has the following *Cobb–Douglas* production function for a certain product:

$$P(x, y) = 80x^3y, \tag{1}$$

where x is the labor, measured in dollars, and y is the capital, measured in dollars. Suppose that the company can make a total investment in labor and capital of \$100. How should it allocate the investment between labor and capital in order to maximize the production P?

**Note: Eq.*(1) *is not a standard Cobb–Douglas model; it has been significantly simplified for easier computation.*

20. (13 marks) Consider

$$\begin{cases} u &= f(ux, v + y), \\ v &= g\left(u - x, v^2 y\right), \end{cases}$$

where $f = f(\alpha, \beta)$ and $g = g(\gamma, \delta)$ are continuous functions; the following partial derivatives exist,

$$f_\alpha \equiv \frac{\partial f}{\partial \alpha}, \qquad f_\beta \equiv \frac{\partial f}{\partial \beta}, \qquad g_\gamma \equiv \frac{\partial g}{\partial \gamma}, \qquad g_\delta \equiv \frac{\partial g}{\partial \delta}$$

Try to find the derivatives

$$\frac{\partial u}{\partial x} \qquad \text{and} \qquad \frac{\partial v}{\partial x}.$$

*[Hint] Recommended computational approach: *doing differentials for both sides of the equations*.

Glossary

Capital	本金，资本
Cartesian	笛卡尔
Classify	分类，归类
Compression	压缩
Critical point	临界点
Decreasing	减少
Directional derivative	方向导数
Ellipsoid	椭球面
Error	误差
Heat Index	热指数
Humidity	湿度
Hyperboloid	双曲面
Increasing	增加
Increment	增量
Labor	劳动力
Laplacian operator	拉普拉斯算子
Maximum	极大
Measurement	测量
Metal	金属
Minimum	极小
Normal	垂直
Paraboloid	抛物面
Parallel	平行
Partial derivative operator	偏导算子
Perpendicular	垂直的
Planar curve	平面曲线
Polar coordinates	极坐标
Production	产值
Quadratic surface	二次曲面
Reflection	反射
Rotate	旋转
Saddle point	鞍点
Sphere	球
String parameter	弦参数
Surface	曲面
Tangent	相切
Temperature field	温度场
Total differential	全微分
Unique	唯一
Variance	变化量