Beijing-Dublin International College

SPRING TRIMESTER EXAMINATION - 2023/2024

MATH1002J Introduction to Analysis

Prof. Nial Friel (Head of School)
Dr. Daniele Casazza

Time Allowed: 120 minutes

Instructions for Candidates

For full marks you must **answer ALL questions** and show clearly **all the steps and computations** in your answer (each question is worth 20 marks).

You are not allowed to use lecture notes or any material beyond a pen (no pencil), a non programmable calculator, and a ruler (if needed).

BJUT Student ID: _____ **UCD Student ID:** _____

I have read and clearly understand the Examination Rules of both Beijing University of Technology and University College Dublin. I am aware of the Punishment for Violating the Rules of Beijing University of Technology and/or University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I accept the punishment thereof.

Honesty Pledge: _____ **(Signature)**

Instructions for Invigilators

This examination is closed-book and closed-notes. Non-programmable calculators are permitted. No rough-work paper is to be provided to the candidate.

1. (a) Compute the radius of convergence of $a(x) = \sum_{n\geq 1} n^6 x^n$. [4]

(b) Compute the radius of convergence of $b(x) = \sum_{n\geq 1} (-e)^{-n^2} x^n$. [4]

(c) Show that the power series $c(x) = \sum_{n\geq 0} (\sin n) \cdot x^n$ is absolutely convergent for $|x| < 1$, so that its radius of convergence is $R \geq 1$. [4]

(d) Show that there are infinitely many n for which $|\sin n| \geq 1/2$ (if you need, you can use the trigonometric formulas $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$ and/or $\sin^2 x + \cos^2 x = 1$...). Conclude that $\sin n$ does not converge to 0 when $n \to \infty$. [4]

(e) Combine parts 1(c) and 1(d) to show that the radius of convergence of $\sum_{n\geq 0} (\sin n) \cdot x^n$ is $R = 1$. [4]

2. Decide if the following statements are True or False, and justify your claims.

(a) If a statement $\mathcal{P}(n)$ is true for all $n \geq 1$ except only at the single value $n = 5$, we can still prove it by induction from the base $n = 1$. [4]

(b) If $a(x) = \sum_{n\geq 1} a_n x^n$ and $b(x) = \sum_{n\geq 1} b_n x^n$ are two power series that are both divergent at $x = 4$, $a(x) + b(x)$ is divergent at $x = 4$ too. [4]

(c) A sequence a_n may have two different limits as $n \to \infty$. [4]

(d) If 8 divides the integer n^2, then 8 divides n. [4]

(e) If $f : \mathbb{R} \to \mathbb{R}$ is not continuous at $x = 0$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous, and not the zero function ($g \neq 0$, i.e. there exists some x_0 such that $g(x_0) \neq 0$), then the product $f(x)g(x)$ is never continuous at $x = 0$. [4]

3. Consider the function $f : [0,1] \to \mathbb{R}$ defined by the following conditions:

$$f(x) = \begin{cases} \frac{1}{2^{n+1}} + \sqrt{\frac{1}{2^{2n+2}} - \left(x - \frac{1}{2^n}\right)^2}, & \text{if } x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \text{ for some } n \in \mathbb{N} \cup \{0\} \\ 0, & \text{if } x = 0 \end{cases}$$

(a) Sketch approximately the graph of the function for $x \geq \frac{1}{8}$. [5]

(b) Show that the function is continuous for all $x > 0$. [10]

(c) Show that the function is not differentiable everywhere in $(0,1)$. [5]

(BONUS) Is the function continuous at $x = 0$? [+5]

4. (a) Prove that the function:

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

is differentiable everywhere, but its derivative is not continuous. [10]

(b) Use the definition of limit of a sequence to show that the sequence $a_n = \sqrt{1 + \frac{1}{n}}$ converges (justify all the steps). [5]

(c) You have five distinct real numbers $\{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\} \subset \mathbb{R}$. Using the pigeonhole principle, show that there are two distinct elements α and β in the list such that $\sin\alpha \sin\beta > 0$ and $\cos\alpha \cos\beta > 0$. [5]

5. Consider a power series $y(x) = \sum_{n\geq 0} a_n x^n$.

(a) Write the conditions that the coefficients need to satisfy if $y(x)$ is a solution of the differential equation $y''(x) = 2y'(x)$. [5]

(b) Using the proof method that you prefer, find the general expression of a_n in terms of a_1 for all $n \geq 1$, and write the solution as a power series. [5]

(c) Show that the solution found in part 5(b) is a power series with radius of convergence $R = +\infty$. [5]

(d) Describe the solution of part 5(b) it in terms of elementary functions. [5]

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