



Beijing-Dublin International College



SEMESTER II FINAL EXAMINATION – 2022/2023

**Beijing-Dublin International College
BDIC1042J Maths 3 (Advanced Mathematics; Finance)**

PRINCIPAL OF COLLEGE: Wenying WU
MODULE LECTURER: Xin LIU

Time Allowed: 95 minutes

Instructions for Candidates

Your answers to ALL the questions should be written on the Answer Book.

BJUT Student ID: _____ **UCD Student ID:** _____

I have read and clearly understand the Examination Rules of both Beijing University of Technology and University College Dublin. I am aware of the Punishment for Violating the Rules of Beijing University of Technology and/or University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I accept the punishment thereof.

Honesty Pledge: _____ **(Signature)**

Instructions for Invigilators

No calculators are permitted. NO dictionaries are permitted.
No rough-work paper is to be provided for candidates.

NOTE: The exam paper has **3** sections on **8** pages, with a full score of 100 marks.

SECTION A — MULTIPLE CHOICE QUESTIONS

In each question, choose **at most one** option.

Write your answers on the **Answer Book** provided.

This section is worth a total of **15** marks, with each question worth **3** marks.

1. How to obtain the shape of the following paraboloid in space?

$$z = 4x^2 + y^2$$

- (a) Step 1: Starting from a planar curve $z = x^2$.

Step 2: Rotating about the z -axis to obtain: $z = x^2 + y^2$.

Step 3: Doing compression in the x -direction by $x \rightarrow 2x$, to obtain: $z = (2x)^2 + y^2$.

- (b) Step 1: Starting from a planar curve $z = x^2$.

Step 2: Rotating about the x -axis to obtain: $z = x^2 + y^2$.

Step 3: Doing compression in the x -direction by $x \rightarrow 2x$, to obtain: $z = (2x)^2 + y^2$.

- (c) Step 1: Starting from $z = 4x^2 - y^2$.

Step 2: Doing reflection by $y \rightarrow -y$, to obtain: $z = 4x^2 + y^2$.

- (d) Step 1: Starting from a plane $z = 2x + y$.

Step 2: Rotating about the line $x + y + z = 0$ to obtain: $z = (2x)^2 + y^2$.

2. On a two dimensional plane endowed with the Cartesian coordinates (x, y) , let $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ be the partial derivative operators with respect to x and y , respectively.

Which of the following computations is correct?

(a) $\frac{\partial}{\partial x} \left(\frac{y}{x} \frac{\partial}{\partial y} \right) = \frac{y}{x} \frac{\partial^2}{\partial x \partial y}$.

(b) $\frac{\partial}{\partial x} \left(\frac{y}{x} \frac{\partial}{\partial y} \right) = \frac{y}{x} \frac{\partial^2}{\partial y \partial x} - \frac{y}{x^2} \frac{\partial}{\partial y}$.

(c) $\frac{\partial}{\partial y} \left(\frac{y}{x} \frac{\partial}{\partial x} \right) = \frac{1}{x} \frac{\partial^2}{\partial y \partial x}$.

(d) $\frac{\partial}{\partial y} \left(\frac{y}{x} \frac{\partial}{\partial x} \right) = \frac{y}{x} \frac{\partial^2}{\partial y \partial x} - \frac{y}{x^2} \frac{\partial}{\partial y}$.

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3. In terms of the two-dimensional Cartesian coordinate system, (x, y) , the Laplacian operator is given by $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Introducing the polar coordinate system (ρ, θ) through the transformation

$$\begin{cases} x = \rho \cos \theta, \\ y = \rho \sin \theta, \end{cases} \quad \text{i.e.,} \quad \begin{cases} \rho = \sqrt{x^2 + y^2}, \\ \theta = \arctan \frac{y}{x}, \end{cases}$$

one can express the Laplacian operator Δ in terms of the ρ and θ coordinates as _____.

- (a) $\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2}$.
 (b) $\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \theta^2}$.
 (c) $\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial}{\partial \rho} + \frac{1}{\rho} \frac{\partial^2}{\partial \theta^2}$.
 (d) $\Delta = \frac{1}{\theta} \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial^2}{\partial \theta^2}$.

4. The following quadratic surface is a _____:

$$x^2 + 2z^2 - 6x - y + 10 = 0.$$

- (a) sphere; (b) ellipsoid; (c) hyperboloid; (d) paraboloid.

5. The total differential of a two-variable function $z = (x^2 - y^2) e^y$ is given by

- (a) $dx + dy = (x^2 - y^2) e^y dz$;
 (b) $dz = (x^2 - y^2 + 2x - 2y) [e^y dx + (x^2 - y^2) dy]$;
 (c) $dz = e^y [2x dx + (x^2 - y^2 - 2y) dy]$;
 (d) $dz = 2xe^y dx + (x^2 - y^2 - 2y) dy$.

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SECTION B — GAP-FILLING QUESTIONS

Write your answers on the **Answer Book** provided.

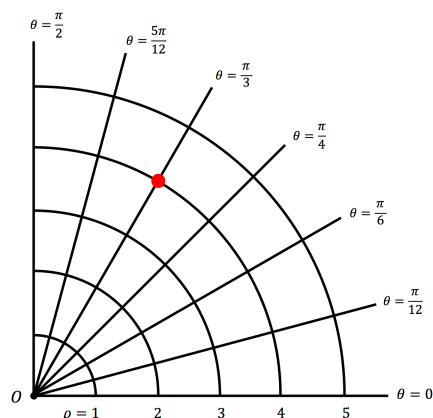
This section is worth a total of **20** marks, with each question worth **4** marks.

6. Let C be a curve in space

$$C : \begin{cases} x + y^2 - z^2 = 0, \\ x + y - 2z = 0. \end{cases}$$

The equation of the projection of C onto the xz -plane is _____.

7. Consider a two-dimensional plane endowed with the polar coordinate system (ρ, θ) , with ρ expressed in meters (m) and θ in radian (rad, dimensionless). Let $T = T(\rho, \theta)$ (expressed in Celsius, °C) be a temperature field distributed on the plane, as shown in the figure and table below.



Temperature (T) distribution as a function of radius ρ and angle θ
In two-dimensional polar coordinate system (ρ, θ)

$\rho \backslash \theta$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
0	96	98	100	103	106	109	112
1	100	103	105	108	112	115	119
2	104	107	111	114	117	122	127
3	109	113	116	121	125	129	135
4	114	118	123	126	132	138	149
5	119	124	129	135	139	147	154

Try to find the following partial derivatives at the point $(\rho, \theta) = (4, \frac{\pi}{3})$:

$$\frac{\partial T}{\partial \rho} \approx \frac{\Delta T}{\Delta \rho} = \text{_____}, \quad \frac{\partial T}{\partial \theta} \approx \frac{\Delta T}{\Delta \theta} = \text{_____},$$

where $\Delta \rho$ and $\Delta \theta$ are the minimum variance of ρ and θ , respectively, near the studied point.

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8. Evaluate the limit:

$$\lim_{(x,y) \rightarrow (0,0)} x^2 y^2 \ln(x^2 + y^2) = \underline{\hspace{2cm}}.$$

9. Let $f = axy$, where a is a constant. y is a function depending on x through the relation

$$x^2 + a^2 = y^2.$$

Evaluate $\frac{df}{dx} = \underline{\hspace{2cm}}.$

10. Consider a function $z = f(x, y)$ and the following three vectors in the xy -plane,

$$\mathbf{u}_1 = \mathbf{i} + 2\mathbf{j}, \quad \mathbf{u}_2 = 2\mathbf{i} + 2\mathbf{j}, \quad \mathbf{u}_3 = 3\mathbf{i} - 4\mathbf{j}.$$

If the directional derivatives in $\mathbf{u}_1$ and $\mathbf{u}_2$ are $D_{\hat{\mathbf{u}}_1} f = -\frac{1}{\sqrt{5}}$ and $D_{\hat{\mathbf{u}}_2} f = 0$, respectively, try to compute the directional derivative along $\mathbf{u}_3$: $D_{\hat{\mathbf{u}}_3} f = \underline{\hspace{2cm}}.$

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SECTION C — EXTENDED ANSWER QUESTIONS

Write your answers on the **Answer Book** provided.

This section is worth a total of **65** marks. The marks of each question are as shown.

- 11. (8 marks)** Consider a hill in a shape described by the function

$$z = f(x, y) = x^2y^3 - 4y.$$

Let $P(2, -1, 0)$ be a point on the hill.

- (a) Find the fastest increasing direction at P , denoted as $\hat{l}_1$, and the directional derivative in that direction, denoted as $D_{\hat{l}_1}f$.
- (b) Find the fastest decreasing direction at P , denoted as $\hat{l}_2$, and the directional derivative in that direction, denoted as $D_{\hat{l}_2}f$.
- (c) Find the direction $\hat{l}_3$ at P along which the directional derivative has zero change.

- 12. (8 marks)** Classify every critical point of a function

$$z = f(x, y) = x^4 + y^4 - 4xy + 1, \quad x, y \in \mathbb{R}$$

to be a local maximum/minimum point or a saddle point.

BDIC1042J Advanced Mathematics (Module 3; Finance)**13. (10 marks)** *Black–Scholes–Merton model (Nobel Prize in Economic Sciences, 1997):*

This is a mathematical model of a financial market containing derivative investment instruments. From the model, one can deduce the Black–Scholes formula which gives a theoretical estimate of the price of *European-style options*. The formula leads to a boom in options trading and provided mathematical legitimacy to the activities of the *Chicago Board Options Exchange* and other options markets around the world. It is widely used, although often with adjustments and corrections, by options market participants. Many empirical tests have shown that the Black–Scholes price is “fairly close” to the observed prices (*although there are well-known discrepancies such as the “option smile”).

The Black–Scholes formula reads

$$\begin{aligned} C(S, t) &= N(d_1)S - N(d_2)Ke^{-rt}, \\ P(S, t) &= -N(-d_1)S + N(-d_2)Ke^{-rt}, \end{aligned}$$

where,

$C(S, t)$ — price of an European call option;

$P(S, t)$ — price of an European put option;

K — strike price of the option;

r — annualized risk-free interest rate, continuously compounded (the force of interest);

t — time left before maturity (expressed in years);

$N(d_1)$, $N(d_2)$ — probabilities of the option expiring in-the-money (standard normal distributions).

Suppose we have a setting:

When $d_1 = 0.8$ and $d_2 = 0.6$, there are

$$N(d_1) = 0.8, \quad N(-d_1) = 0.2; \quad N(d_2) = 0.7, \quad N(-d_2) = 0.3.$$

Taking $S = 42$, $K = 40$, $r = 0.1$ and $t = 0.5$, we have $Ke^{-rt} = 40$.

Then, if S has an increment $\Delta S = 1$ and t an increment $\Delta t = -0.01$, try to compute:

- (a) the increment of the price $C(S, t)$, namely, ΔC ;
- (b) the increment of the price $P(S, t)$, namely, ΔP .

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- 14. (11 marks)** *Cobb-Douglas economical model* (1922): Suppose the total production P of a certain product is given by

$$P = bL^\alpha K^{1-\alpha},$$

where b and α are constants, with $b > 0$ and $0 < \alpha < 1$. The amount L denotes how many units of labor to be used, and K how many units of capital to be invested for producing this product. Then the total cost is given by

$$C = mL + nK,$$

where m is the cost of a unit of labor, and n the cost of a unit of capital. m and n are positive constants.

- (a) If the budget for the total cost C is fixed, find the values of L and K to maximize the total production P .
- (b) If the total production P is fixed, find the values of L and K to minimize the total cost C .

- 15. (13 marks)** Given

$$\begin{cases} u = f(ux, v + y), \\ v = g(u - x, v^2 y), \end{cases}$$

where $f = f(\alpha, \beta)$ and $g = g(\gamma, \delta)$ are two continuous functions, and the partial derivatives

$$f_\alpha \equiv \frac{\partial f}{\partial \alpha}, \quad f_\beta \equiv \frac{\partial f}{\partial \beta}, \quad g_\gamma \equiv \frac{\partial g}{\partial \gamma}, \quad g_\delta \equiv \frac{\partial g}{\partial \delta}$$

exist. Then, try to find the derivatives

$$\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial x}.$$

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16. (15 marks)

In the 3-dimensional space there is a flyscreen lying in a plane Σ ,

$$\Sigma : 2x + y - 2z = 1.$$

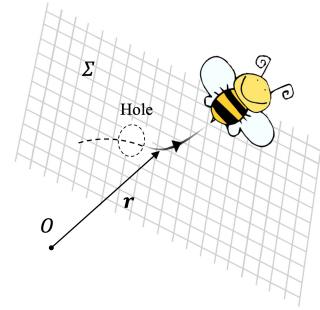
Let $\mathbf{r}$ denote the trajectory of a flying bee, as a function of time t :

$$\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, \quad t \geq 0.$$

- (a) Find all the time t when the bee is flying parallel to Σ .
- (b) Find all the time t when the bee is flying perpendicular to Σ .
- (c) There are holes in the screen Σ through which the bee passes.

Find the coordinates of the holes.

(Hint: For Item (c), you may need to do polynomial factorization, in which $(t + 1)$ is a factor.)



Glossary

Budget	预算
Call option	看涨期权
Capital	本金，资本
Cartesian	笛卡尔
Classify	分类，归类
Compression	压缩
Cost	投资
Critical point	临界点
Decreasing	减少
Directional derivative	方向导数
Ellipsoid	椭球面
Endowed with	赋予
Flyscreen	防虫网
Hyperboloid	双曲面
Increasing	增加
Increment	增量
Labor	劳动力
Laplacian operator	拉普拉斯算子
Maturity	到期，成熟
Maximum	极大
Minimum	极小
Paraboloid	抛物面
Parallel	平行
Partial derivative operator	偏导算子
Perpendicular	垂直的
Planar curve	平面曲线
Polar coordinates	极坐标
Polynomial factorization	多项式的因式分解
Probabilities	概率
Production	产量
Projection	投影
Put option	看跌期权
Quadratic surface	二次曲面
Radian	弧度
Reflection	反射
Rotate	旋转
Saddle point	鞍点
Sphere	球
Strike price of the option	期权执行价格，又称协议价格
Temperature field	温度场
Total differential	全微分
Trajectory	轨迹
Variance	变化量