



北京工业大学北京 – 都柏林国际学院  
BEIJING-DUBLIN INTERNATIONAL COLLEGE AT BJUT

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**SEMESTER 2 RESIT EXAMINATIONS**  
**ACADEMIC YEAR 2022/2023**

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**EEEN3004J Digital Signal Processing**

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I have read and clearly understand the Examination Rules of both Beijing University of Technology and University College Dublin. I am aware of the Punishment for Violating the Rules of Beijing University of Technology and/or University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I accept the punishment thereof.

**Honesty Pledge:** \_\_\_\_\_ **(Signature)**

**Time Allowed: 2 HOURS**

**Instructions for Candidates**

This examination paper contains **FOUR** questions.

Answer any **THREE** questions for a maximum of 300 marks.

All four questions carry equal marks.

The distribution of marks for the parts of each question is indicated in ( ) brackets

**Instructions for Invigilators**

Non-programmable calculators are permitted.

No rough-work paper is to be provided for candidates.

## Formulas

These formulas are provided without further context.

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{-j \frac{n 2\pi t}{T}}$$

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} T/2 f(t) e^{-j \frac{n 2\pi t}{T}} dt$$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$$

$$F(z) = \sum_{n=-\infty}^{\infty} f_n z^{-n}$$

$$h(t) = f * g = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau$$

$$h(t) = f * g \Leftrightarrow H(j\omega) = F(j\omega)G(j\omega)$$

$$f(t) = \begin{cases} 1, & \text{if } |t| < \frac{L}{2} \\ 0, & \text{otherwise} \end{cases} \Leftrightarrow F(j\omega) = L \operatorname{sinc}\left(\frac{\omega L}{2}\right) = L \frac{\sin\left(\frac{\pi \omega L/2}{2}\right)}{\frac{\pi \omega L}{2}}$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$y_n = \sum_{k=0}^M b_k x_{n-k} - \sum_{l=1}^L a_l y_{n-l}$$

$$\sum_{n=0}^{N-1} e^{j \frac{2\pi n k}{N}} = \begin{cases} N, & \text{if } k \text{ is an integer multiple of } N \\ 0, & \text{otherwise} \end{cases}$$

$$\cos(60^\circ) = \frac{1}{2}$$

$$\sin(60^\circ) = \frac{\sqrt{3}}{2}$$

The variance of a uniform random variable on  $\pm\Delta/2$  is  $\Delta^2/12$

**Q 1**

- a. Consider the filter given by the following difference equation.

$$y_n = y_{n-1} + x_n - x_{n-2} \quad (1)$$

Is this an IIR or an FIR filter? Justify your answer. (10 marks)

- b. Calculate the transfer function of this filter. (15 marks)

- c. Find the filter's poles and zeros. Draw its pole-zero plot. (15 marks)

- d. Calculate the impulse response of this filter. Comment on the result. (30 marks)

- e. Sketch the amplitude response of the filter, determined by any means you care to employ. Show your work. (15 marks)

- f. This filter is to be cascaded with a second filter that has the following transfer function.

$$\tilde{H}_2(z) = \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)} \quad (2)$$

Under what conditions does the combined filter have linear phase? (15 marks)

**(Total: 100 marks)**

**Q 2**

In Fig. 1, we have an extract of a famous FFT paper in which an algorithm was introduced for a prime length signal. Read that extract and answer the questions below.

- a. Identify the equation in which Rader defines the DFT. His notation is slightly different from that we used in class, so define all of the terms. (10 marks)

- b. Write Equation 7 in matrix form for the  $N = 5$  case, given the fact that 2 is a primitive root of 5. (20 marks)

$$A_k = \sum_{i=0}^{N-1} a_i \exp(-j(2\pi/N)ik). \quad (1)$$

The values of  $A_k$  are samples of the z-transform of the finite length sequence  $\{a_i\}$  at  $N$  points equally spaced on the unit circle.

We need a notation for an integer modulo  $N$ . For brevity we will indicate this by superfluous parentheses:

$$((X)) = X \text{ modulo } N. \quad (2)$$

If  $N$  is prime there is some number  $g$ , not necessarily unique, such that there is a one-to-one mapping of the integers  $i=1, \dots, N-1$  to the integers  $j=1, 2, \dots, N-1$ , given by

$$j = ((g^i)). \quad (3)$$

For example, let  $N=7$  and  $g=3$ . The table below gives the mapping of  $i$  onto  $j$ .

|     |   |   |   |   |   |   |
|-----|---|---|---|---|---|---|
| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
| $j$ | 3 | 2 | 6 | 4 | 5 | 1 |

In number theory,  $g$  is called a primitive root of  $N$ . A table of primitive roots of primes less than  $10^4$  is given by Abramowitz and Stegun.<sup>[3]</sup>

## II. COMPUTATION OF $\{A_k\}$ WHEN $N$ IS PRIME

In (1) we have an expression for  $A_k$  for all  $k$ . The expression for  $A_0$  is particularly simple,

$$A_0 = \sum_{i=0}^{N-1} a_i, \quad (4)$$

and is to be computed directly. For the other  $A_k$  we observe that  $a_0$  is not to be multiplied, and we choose to add it last into the summation. We are left with the sequence  $\{A_k - a_0\}$ ,  $k=1, 2, \dots, N-1$  to compute, given by

$$A_k - a_0 = \sum_{i=1}^{N-1} a_i \exp\left(-j \frac{2\pi}{N} ik\right). \quad (5)$$

We permute the terms in the summation, and change the order of the equations via transformations

$$\begin{aligned} i &\rightarrow ((g^i)) \\ k &\rightarrow ((g^k)) \end{aligned} \quad (6)$$

and, noting that  $((g^{N-1})) = ((g^0))$ , we can see that

$$(A_{((g^k))} - a_0) = \sum_{i=0}^{N-1} a_{((g^i))} \exp\left(-j \frac{2\pi}{N} g^{(i+k)}\right). \quad (7)$$

We are now able to recognize that the sequence  $\{A_{((g^k))} - a_0\}$  is the circular correlation of the sequence  $\{a_{((g^i))}\}$  and the sequence  $\{\exp(-j(2\pi/N)g^i)\}$ . But Stockham has shown how circular (or ordinary) correlation functions can be computed with a greatly reduced number of operations by making use of FFT algorithms.<sup>[4]</sup> There are two ways we can proceed. Since  $N$  is prime,  $N-1$  must be composite. Suppose that it is highly composite. Then the  $N-1$  point circular correlation (7) may be recognized as the inverse DFT of the product of the DFT of  $\{a_{((g^i))}\}$  and the DFT of  $\{\exp(-j(2\pi/N)g^i)\}$ .<sup>1</sup> All the DFT operations called for are performed by an FFT algorithm.

$$\{A_{((g^k))} - a_0\} = \text{DFT}^{-1}\left\{\left(\text{DFT}\{a_{((g^i))}\}\right)\left(\text{DFT}\left\{\exp\left(-j \frac{2\pi}{N} g^i\right)\right\}\right)\right\}. \quad (8)$$

<sup>1</sup> It may be shown that this DFT has magnitude  $\sqrt{N}$  at all frequencies but the zeroth, where it is  $-1$ .

Figure 1: Extract of Rader, C.M., 1968. "Discrete Fourier transforms when the number of data samples is prime." Proceedings of the IEEE, 56(6), pp.1107-1108.

**c.** A signal is sampled at 8 kHz, yielding the following samples:  $\{0.2 + j0.2, -0.0442 - j0.0442, -0.1284 - j0.1284, 0.252 + j0.252, -0.2794 - j0.2794\}$ .

- i Calculate the DFT of the signal using Rader's algorithm and length 4 DFTs.
- ii Confirm that your answer is correct by performing the same calculation with the 5-point DFT.

Show your work. (60 marks)

**d.** Plot the amplitude and phase of your answer. Given that the sampling period in the signal is 8 kHz, label your axes carefully. (10 marks)

**(Total: 100 marks)**

### Q 3

**a.** Define the Dirac delta function,  $\delta(t)$ , in terms of a limit. (15 marks)

**b.** State the sifting property of the Dirac delta function. (15 marks)

**c.** Hence or otherwise, calculate the Fourier transform of a Dirac delta function. Show your work. (15 marks)

**d.** Prove the following equivalence.

$$\delta(f) = 2\pi\delta(\omega) \quad (3)$$

(15 marks)

**e.** Prove that the Fourier series of a Dirac delta train,  $\delta_T(t)$ , is given by the following expression.

$$\delta_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \quad (4)$$

Don't forget to define  $\omega_s$ . (20 marks)

f. Consider a signal,  $f(t)$ , that is sampled by a Dirac delta train:

$$\bar{f}(t) = f(t)\delta_T(t) \quad (5)$$

Calculate the spectrum of  $\bar{f}(t)$ . (20 marks)

**(Total: 100 marks)**

## Q 4

a. Stabilise the following IIR filter by cascading it with an all-pass filter.

$$\tilde{H}(z) = \frac{(z - 0.5)^2}{(z - 2(1 + j))(z - 2(1 - j))} \quad (6)$$

(30 marks)

b. Calculate the first 8 outputs of the filter given input samples,  $\{f_n\} = \{\frac{3}{8}, -\frac{1}{4\sqrt{2}} - j0.0518, -\frac{1}{8} + j\frac{1}{4}, \frac{1}{4\sqrt{2}} - j0.3018, -\frac{1}{8}, \frac{1}{4\sqrt{2}} + j0.3018, -\frac{1}{8} - j\frac{1}{4}, -\frac{1}{4\sqrt{2}} + j0.0518\}$ . You can assume that  $f_n = 0 \forall n < 0$ .

If you couldn't answer part a, calculating the output of the filter without stabilisation is sufficient to obtain full marks. (30 marks)

c. Use 1 stage of the radix-2 decimation-in-time FFT algorithm that we studied in class to split an 8-point DFT into 2 x 4-point DFTs, and calculate the DFT of the filter input. (40 marks)

**(Total: 100 marks)**

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