



An Coláiste Ollscoile, Baile Átha Cliath
University College Dublin

SPRING, 24/25
TRIMESTER EXAMINATIONS

EEEN3018J
Numerical Methods and Optimisation

Module Coordinator: Dr Declan Delaney and Dr John Healy

Student Number

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Seat Number

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Time Allowed: 120 minutes

Exam materials to be supplied:
12 page answer booklets

Instructions to students:

Answer **Question 1, AND EITHER Question 2 or Question 3, AND EITHER Question 4 or Question 5**. The percentages in the right margin give an approximate indication of the relative importance of each part of the question.

Formulas

These formulas are provided without further context.

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})}$$

$$\begin{pmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & n \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{pmatrix}$$

$$A^T A c = A^T y$$

$$a_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

$$b_i = y_i$$

$$\begin{bmatrix} \frac{2h_0}{3} & \frac{h_0}{3} & 0 & \dots & 0 & 0 & 0 \\ \frac{h_0}{3} & \frac{2}{3}(h_1 + h_0) & \frac{h_1}{3} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{h_{n-2}}{3} & \frac{2}{3}(h_{n-2} + h_{n-1}) & \frac{h_n}{3} \\ 0 & 0 & 0 & \dots & 0 & \frac{h_{n-1}}{3} & \frac{2h_{n-1}}{3} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} \frac{f_0}{h_0} - \alpha \\ \frac{f_1}{h_1} - \frac{f_0}{h_0} \\ \vdots \\ \frac{f_{n-1}}{h_{n-1}} - \frac{f_{n-2}}{h_{n-2}} \\ \beta - \frac{f_{n-1}}{h_{n-1}} \end{bmatrix}$$

$$hf\left(\frac{h}{2}\right)$$

$$h[f(0) + f(h)]$$

$$\frac{h}{3}[f(0) + 4f(h) + f(2h)]$$

$$x_{n+1} = x(t_n) + \Delta t f(t_n, x_n)$$

$$x_{n+1} = x_n + \frac{1}{2}\Delta t [f(t_n, x_n) + f(t_n + \Delta t, x_{n+1}^p)]$$

$$k_1 = \Delta t f(t_n, x_n)$$

$$k_2 = \Delta t f(t_n + \alpha \Delta t, x_n + \beta k_1)$$

$$x_{n+1} = x_n + ak_1 + bk_2$$

Q 1

a. Convert the following program to Standard Equality Form:

$$\begin{aligned}
 \max \quad & z(x) = 3x_1 + 2x_2 - x_3 + x_4 \\
 \text{s.t.} \quad & x_1 + 2x_2 + x_3 - x_4 \geq 5 \\
 & -2x_1 - 4x_2 + x_3 + x_4 \leq -1 \\
 & x_2 + 2x_3 = -9 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

(10 marks)

b. Consider the LP problem (P):

$$\max\{c^T x : Ax = b, x \geq 0\}, \text{ where}$$

$$A = \begin{pmatrix} 5 & 3 & -1 & 0 \\ -2 & -4 & 2 & 9 \\ 2.5 & 6 & -7 & 5 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 13 \\ 24 \end{pmatrix} \quad c^T = (2 \quad 9 \quad 4 \quad -7)$$

write an auxiliary problem (A) for the LP problem with a known feasible Basis B for A .
(10 marks)

c. A farmer is planning the seasonal cultivation of wheat and corn. Each acre of wheat requires 3 units of fertilizer and 2 hours of labour. Each acre of corn requires 4 units of fertilizer and 4 hours of labour. Fertilizer costs CYN 10 per unit and labour costs CYN 15 per hour. Wheat can be sold for CYN 00 per acre, while corn sells for CYN 350 per acre.

The farmer has a maximum of 300 hours of labour available and a total budget of CYN 3,000 to spend on fertilizer and labour combined.

Although corn is more profitable, wheat plays an important role in preventing soil erosion. As a result, wheat must constitute at least 40% of the total cultivated area.

Formulate a linear programming model to maximise the farmer's net profit (i.e. total revenue minus input costs). Clearly define your decision variables, the objective function, and all constraints based on the given information.

(10 marks)

- d. Use the graphical approach to solve the following LP.

$$\begin{aligned} \max \quad & z(x) = x_1 + 10x_2 \\ \text{s.t.} \quad & 3x_1 + 5x_2 \leq 15 \\ & -2x_1 + x_2 \geq -2 \\ & 1.5x_1 + x_2 \leq 6 \\ & x_1, x_2 \leq 0 \end{aligned}$$

What is the optimal solution x^* and optimal value for the objective function?
(10 marks)

- e. Describe, in your own words, the certificate of optimality for a Linear Program.
(10 marks)

(Total: 50 marks)

Q 2

Consider the LP problem (P):

$$\max\{c^T x : Ax = b, x \geq 0\}, \text{ where}$$

$$A = \begin{pmatrix} 1 & -3 & 5 & 0 \\ 0 & 3/2 & 1 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 15 \\ 6 \end{pmatrix} \quad c^T = (0, 1, 10, 0)$$

Use the simplex algorithm to find the optimal solution x^* and optimal value $z(x^*)$ for the problem.

- Use Bland's rule to decide new entries to the Basis set.

(Total: 50 marks)

Q 3

- a. For the following function:

$$f(x, y, z) = 2x^2 + 4y^2 + z^2 + 2xy + 5yz - 4x + 14y + 20z - 7$$

Determine whether the function is convex.

Hint: Determine whether the Hessian matrix (H) is positive definite. (10 marks)

b. For the following function:

$$f(x) = 2x^2 + 3x - 15$$

and an initial region of uncertainty $[a, b] = [-4, 2]$, how many steps are required to determine the optimal value within a tolerance of .002 for the golden search?

How many function evaluations are required throughout the process? (10 marks)

c. For the following function:

$$f(x) = 2x^4 - 3x^3 + 20x^2 - 10x + 2$$

and an initial region of uncertainty $[a, b] = [-2, 3]$, determine the new region of uncertainty after each step of the golden search algorithm? complete the algorithm for 2 steps. (30 marks)

(Total: 50 marks)

Q 4

a. You are measuring the D.C. voltages at three nodes in an electronic circuit: V_1 , V_2 , and V_3 . Each is intended to be measured independently, but the measurement channels are poorly isolated due to a flawed design. As a result, each measured voltage includes leakage from other channels. This has been characterised experimentally as:

$$b_1 = V_1 + 0.999 V_2, \quad b_2 = V_1 + V_2, \quad b_3 = V_2 + V_3$$

where b_1 , b_2 , and b_3 are voltage probe readings recorded by a poorly shielded three-channel oscilloscope.

The circuit has a switch incorporated in it. With the switch open, we measured the following values:

$$b_1 = 4.0, \quad b_2 = 4.01, \quad b_3 = 6.0$$

With the switch closed, we measured as follows:

$$b_1 = 4.0, \quad b_2 = 4.005, \quad b_3 = 6.0$$

- i Write this problem as a pair of systems of equations, $A\vec{v}_{\text{open}} = \vec{b}_{\text{open}}$ and $A\vec{v}_{\text{closed}} = \vec{b}_{\text{closed}}$.
- ii Obtain the LU decomposition of A .
- iii Hence solve for $\vec{v}_{\text{open}}$ and $\vec{v}_{\text{closed}}$.
- iv One engineer in your team believes that the switch clearly alters the behaviour of the circuit. Another says we cannot be certain of that. Given that the singular values of A are 2.1748, 1.1261, and 0.0004083, which engineer do you believe? Justify your answer with reference to the condition number of A .

(80 marks)

b. A resistive touchscreen is modelled as a long thin resistive strip. When a user touches the screen, the contact injects a small distributed current into the strip, represented by a profile $f(x)$. The voltage along the strip, $u(x)$, satisfies the Poisson equation:

$$\frac{d^2}{dx^2}u(x) = f(x), \quad (1)$$

with boundary conditions $u(0) = 0$ and $u'(1) = 0$.

To calculate the resulting voltage profile, the problem is converted (by steps that we won't cover here) into a matrix equation:

$$\begin{pmatrix} K_{11} & K_{21} & \cdots \\ K_{12} & K_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \end{pmatrix}, \quad (2)$$

where

$$K_{ij} = \int_0^1 N'_i(x)N'_j(x) dx, \quad F_i = \int_0^1 N_i(x)f(x) dx,$$

and $N_i(x)$ is the i th test function, with derivative $N'_i(x)$. The voltage is then approximated by $u(x) \approx \sum_i e_i N_i(x)$.

By solving for $u(x)$, the location of the touch can be inferred from the voltage distribution.

Describe which numerical methods may be employed to solve for $u(x)$ given $f(x)$, $N_i(x)$, and $N'_i(x)$. (20 marks)

(Total: 100 marks)

Q 5

The frequency of a quartz crystal oscillator drifts slowly over time due to ageing. This drift can be modelled using an exponential decay:

$$f(t) = f_0 e^{-kt} \quad (3)$$

where:

- $f(t)$ is the oscillator frequency in kHz at time t (in days),
- f_0 is the initial frequency,
- k is a constant associated with the ageing behaviour.

The nominal (initial) frequency of the oscillator is $f_0 = 10.0000$ kHz, and its frequency is monitored over a period of 40 days, as shown in Table 1.

- a.** By taking the natural logarithm of both sides of Eq. 3, express the model in a linear form. Then, using least squares estimation, determine the value of k . (80 marks)
- b.** Using the model you derived, estimate the time (in days) at which the oscillator's frequency will have drifted by 100 parts per million (ppm) from its nominal value. (20 marks)

Table 1: Quartz oscillator ageing over time

t (days)	0	10	20	30	40
$f(t)$ (kHz)	10.0000	9.9997	9.9995	9.9993	9.9991

(Total: 100 marks)