



北京工业大学北京 – 都柏林国际学院

BEIJING-DUBLIN INTERNATIONAL COLLEGE AT BJUT

202/2024 SPRING TRIMESTER EXAMINATIONS

EEEN3018J

Numerical Methods and Optimization

Module Coordinator: Dr Barry Cardiff and Dr. John Healy

Student Number

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Seat Number

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Time Allowed: 90 mins

Materials Permitted in the Exam Venue:

Foreign language dictionary (hard copy)
Non-programmable or scientific calculator
Any one textbook and one folder of notes.

Materials to be Supplied to Students:

12 Page Answer Booklets

Instructions to Students:

Answer EITHER question 1 or 2 AND EITHER question 3 or 4.

All questions carry equal weight. The percentages in the right margin give an approximate indication of the relative weight of each part of a question.

Instructions to Invigilators:

Non-programmable calculators are permitted. No rough-work paper is to be provided for candidates.

PART I: Numerical Methods

Answer either question 1 or 2

Question 1

- 1.1 A function $y(x)$ is known at samples $y(x_i) = y_i$ for $0 < i \leq N$. It is approximated piece-wise by means of functions, $g_i(x)$ such that

$$y(x) \approx g(x) = g_i(x - x_i), \quad x_i < x < x_{i+1}, \quad 0 < n < N - 1$$

Derive expressions for the first order polynomials, $g_i(x)$, in terms of the samples. [25%]

- 1.2 Given the samples of the function $y(x)$ in Table 1, estimate $y(0.5)$ by means of any interpolation method you have learned. You may assume that $y'(0) = y'(3) = 0$ if necessary. Name the interpolation method you used and justify your choice. Show your work. [30%]

x	0	1	2	3
y	2	3	1	0

Table 1: Samples of the function $y(x)$.

- 1.3 Now if a quadratic function is to be fit to the samples in Table 1:

- Derive a matrix equation for the coefficients of the quadratic.
- Solve for the coefficients. Using that polynomial, estimate $y(0.5)$.
- Compare this estimate with the one found in question 1.2.

[45%]

Question 2

- 2.1 (a) Explain the three main components of the IEEE 754 32-bit floating point representation and describe what each component represents.
 (b) How is the special number $-\infty$ represented in that floating point format?
 (c) What is a denormal (or subnormal) number and what is their purpose? [30%]

- 2.2 In the course of designing a square patch antenna constructed with a perfect electrical conductor material, the following equation is obtained:

$$L = \frac{v_0}{2f_r \left(\frac{\epsilon_r + 1}{2} + \frac{\epsilon_r - 1}{2} \sqrt{1 + 12 \frac{h}{L}} \right)}$$

, where L is the patch length, h is the height of the substrate, ϵ_r is the relative dielectric constant of the substrate, f_r is the resonant frequency, and v_0 is the speed of light. All of the values except for L are known.

Outline the steps by which we could solve for L by numerical means, using an algorithm that you have studied in this module. [40%]

2.3 The Fourier transform performs the following integration.

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$$

Given uniform samples of $f(t)$ evaluated at times $t_n = nT$, for $0 \leq n \leq N - 1$, the discrete Fourier transform is given by,

$$\bar{F}_m = \sum_{n=0}^{N-1} f(t_n)e^{-j\omega_m t_n} = \sum_{n=0}^{N-1} f_n e^{-\frac{jnm}{N}}$$

For a given value of m , the DFT is a numerical integration: discuss.

[30%]

PART II: Optimization

Answer either question 3 or 4

Question 3

Consider the function:

$$f(x) = x^3 - 48x + 10 \quad (1)$$

- 3.1 (a) Sketch $f(x)$ on the range $[-8, +8]$
 (b) Is this $f(x)$ linear, prove it?
 (c) Is this $f(x)$ convex?
 (d) Define a range (of x) where in the $f(x)$ is locally convex
 (e) Where does the minimum of $f(x)$ occur within the range $x=[-10, +\infty]$?, and what is the value of $f(x)$ at that point?
 (f) Where does the minimum of $f(x)$ occur within the range $x=[0, +\infty]$?, and what is the value of $f(x)$ at that point? [30%]

- 3.2 Consider using a line search algorithm to find, $x_{\min}$, an estimate of the value of x where the function $f(x)$, in equation (1), is minimized assuming a starting range of $x=[0, +10]$. How many evaluations of $f(x)$ are required for each of the following algorithms to obtain an estimate of $x_{\min}$ with an accuracy of ± 0.01

- (a) Grid search
 (b) Golden search

Explain your logic, and clearly state any assumptions. [40%]

- 3.3 Determine if the following matrix is positive definite

$$\begin{bmatrix} 1 & 3 & 2 \\ -4 & 4 & 4 \\ 1 & -2 & 4 \end{bmatrix}$$

[30%]

Question 4

A company that produces Integrated Circuits (ICs) can make regular CPUs or high performance GPUs. The CPUs sell for a lower price than the GPUs, however the cost of manufacture of each is impacted by several factors, specifically:

- The CPUs and GPUs are cut from large wafers containing many CPUs or GPUs
- GPUs need to run faster so instead of using Silicon (Si) wafers we need to use Gallium Arsenide (GaAs) to make the GPUs.
Thus one wafer can't contain a mixture of both CPUs and GPUs.
- The cost of Si and GaAs wafers are also significantly different.

- The GPUs are bigger than the CPUs meaning that we can get less GPUs per wafer.
- There is a limited supply of wafers available.
- The GPUs are more complex, requiring more processing step meaning the average processing time for each type of wafer is different.

The following tables put numerical values on these issues:

	CPU	GPU
Sale price [\$]	50	120
Wafer type	Si	GaAs
Number of ICs per wafer	70	50

Table 2: Basic IC parameters.

	Wafer cost [\$]	Average processing time [mins]	Number ICs per wafer	Daily availability
Si wafers	1,000	30	70	50
GaAs wafers	2,000	45	50	30

Table 3: Wafer parameters.

The company would like to be able to answer the following questions:

- How many Si and GaAs wafers should it order per day?
- How many CPUs and GPUs should it produce each day?

in order to maximise its daily profit.

Also it would like to know how much profit will it make each day.

Note: We can only process an integer number of wafers per day.

- 4.1 Transform the above natural language problem statement in a mathematical one. Begin by clearly defining all variables, and then clearly writing a compact math maximization problem statement. [30%]
- 4.2 Transform the math formulation in question 4.1 to a problem in Standard Equality Form. [20%]
- 4.3 Find a solution to this problem via any means, i.e. state clearly how many CPUs and GPUs we should make every day and how much profit we'll make per day. [30%]
- 4.4 Now, if the market demands that we produce at least 5000 CPUs per day, how would the SEF derived in question 4.2 to take this into account? [10%]
- 4.5 The weak duality theorem states:
If $\bar{x}$ and $\bar{y}$ are feasible solutions of a primal (P) and dual (D) problems respectively, then the Dual's objective function, $b^T \bar{y}$ is an upper for Primal's objective function, $c^T \bar{x}$, i.e.

$$c^T \bar{x} \leq b^T \bar{y}$$

- Prove the above theorem.

[10%]

These formulas are provided without further context.

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})}$$

$$\begin{pmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & n \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{pmatrix}$$

$$A^T A c = A^T y$$

$$a_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

$$b_i = y_i$$

$$\begin{bmatrix} \frac{2h_0}{3} & \frac{h_0}{3} & 0 & \dots & 0 & 0 & 0 \\ \frac{h_0}{3} & \frac{2}{3}(h_1 + h_0) & \frac{h_1}{3} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{h_{n-2}}{3} & \frac{2}{3}(h_{n-2} + h_{n-1}) & \frac{h_n}{3} \\ 0 & 0 & 0 & \dots & 0 & \frac{h_{n-1}}{3} & \frac{2h_{n-1}}{3} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} \frac{f_0}{h_0} - \alpha \\ \frac{f_1}{h_1} - \frac{f_0}{h_0} \\ \vdots \\ \frac{f_{n-1}}{h_{n-1}} - \frac{f_{n-2}}{h_{n-2}} \\ \beta - \frac{f_{n-1}}{h_{n-1}} \end{bmatrix}$$

$$hf(\frac{h}{2})$$

$$h[f(0) + f(h)]$$

$$\frac{h}{3}[f(0) + 4f(h) + f(2h)]$$

$$x_{n+1} = x(t_n) + \Delta t f(t_n, x_n)$$

$$x_{n+1} = x_n + \frac{1}{2} \Delta t [f(t_n, x_n) + f(t_n + \Delta t, x_{n+1}^p)]$$

$$k_1 = \Delta t f(t_n, x_n)$$

$$k_2 = \Delta t f(t_n + \alpha \Delta t, x_n + \beta k_1)$$

$$x_{n+1} = x_n + a k_1 + b k_2$$

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