北京工业大学北京－都柏林国际学院
BEIJING-DUBLIN INTERNATIONAL COLLEGE AT BJUT

2023/2024 SUMMER TRIMESTER EXAMINATIONS

EEEN3018J

Numerical Methods and Optimization

Module Coordinator: Dr Barry Cardiff and Dr. John Healy

Student Number

Seat Number

Time Allowed: 90 mins

Materials Permitted in the Exam Venue:
Non-programmable or scientific calculator

Materials to be Supplied to Students:
12 Page Answer Booklets

Instructions to Students:
Answer EITHER question 1 or 2 AND EITHER question 3 or 4.
All questions carry equal weight. The percentages in the right margin give an approximate indication of the relative weight of each part of a question.

Instructions to Invigilators:
Non-programmable calculators are permitted. No rough-work paper is to be provided for candidates.

PART I: Numerical Methods

Answer either question 1 or 2

Question 1

1.1 Cubic splines uses N piecewise cubic polynomials for interpolation. Each one is given by:

$$g_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \tag{1}$$

Two constraints are that

$$g_i(x_i) = y_i, \quad \text{and,} \tag{2}$$
$$g_i(x_{i+1}) = y_{i+1} \tag{3}$$

for $i = 0, 1, ..., N - 1$.

- Are these sufficient constraints to define the functions $g_i(x)$?
- If not, what additional constraints or information are required?

[20%]

1.2 From (1) and (2), we find that $d_i = y_i$.
Prove that the a_i and c_i coefficients are given by the following expressions.

$$a_i = \frac{b_{i+1} - b_i}{3h_i}, \quad \text{and,}$$
$$c_i = \frac{f_i}{h_i} - \frac{b_{i+1} + 2b_i}{3}h_i$$

for $i = 0, 1, ..., N - 1$, and where $h_i = x_{i+1} - x_i$ and $f_i = y_{i+1} - y_i$. [40%]

Time	17:00	18:00	19:00	20:00
Temperature (°C)	37.3	36.1	34.1	33.2

Table 1: Samples of the function $y(x)$.

1.3 Table 1 shows temperature data for Beijing one summer's evening. By means of cubic spline interpolation, estimate the temperature at 18:50. You may assume that the derivative at the endpoints is zero. Show your work. [40%]

Question 2

A lithium ion battery has a v-t characteristic curve: its voltage over time. This changes as the battery wears, and can be approximated piece-wise. An important piece of the curve is the rapid discharge stage that occurs as the battery nears the end of its charge cycle. It may be approximated as a decaying exponential as follows.

$$v(t) = V_0 e^{-k(t-t_0)} \tag{4}$$

The battery will be considered totally discharged when the voltage reaches 3 V. Battery management software is monitoring the discharge characteristic of a battery. It has determined that the battery is in rapid discharge (since $t = 478$ minutes) and provided the data shown in Table 2 to a modelling subroutine.

2.1 Use least squares estimation and the data from Table 2 to estimate the unknowns in (4) [70%]

2.2 Estimate the time until the battery is totally discharged. [30%]

t (minutes)	478	481	484	487	490
v (volts)	3.4	3.37	3.34	3.31	3.28

Table 2: V-t characteristic of the battery to be approximated.

Hint: Taking the natural logarithm of both sides of (4) will linearise the problem.

PART II: Optimization

Answer either question 3 or 4

Question 3

Consider the function:

$$f(x) = x^3 - 48x + 10 \tag{5}$$

3.1 (a) Sketch $f(x)$ on the range $[-8, +8]$

(b) Is this $f(x)$ linear, prove it?

(c) Is this $f(x)$ convex?

(d) Define a range (of x) where in the $f(x)$ is locally convex

(e) Where does the minimum of $f(x)$ occur within the range x=[-10,+∞]?, and what is the value of $f(x)$ at that point?

(f) Where does the minimum of $f(x)$ occur within the range x=[0,+∞]?, and what is the value of $f(x)$ at that point?

[30%]

3.2 Consider using a line search algorithm to find, x_{min}, an estimate of the value of x where the function $f(x)$, in equation (5), is minimized assuming a starting range of x=[0,+10].
How many evaluations of $f(x)$ are required for each of the following algorithms to obtain an estimate of x_{min} with an accuracy of ± 0.01

(a) Grid search

(b) Golden search

Explain your logic, and clearly state any assumptions. [40%]

3.3 Determine if the following matrix is positive definite

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

[30%]

Question 4

4.1 Define a Linear Programme using matrix notation. Be sure to clearly state the dimension of all vectors and matrices. [10%]

4.2 Explain the concept of a Feasible region, $\mathcal{X}$. How does it vary with the objective function? [10%]

4.3 What is the Simplex method?, i.e. explain what kind of problems can it solve and what are the possible outcomes from running the simplec method of a particular problem. [10%]

A manufacturing company produces two products, A and B. Each unit of product A requires 2 hours of machining and 3 hours of assembly, while each unit of product B requires 1 hour of machining and 2 hours of assembly. The company has a maximum of 100 hours of machining time and 120 hours of assembly time available per week.
The profit from each unit of product A is €40, and the profit from each unit of product B is €30. The company wants to determine the number of units of each product it should produce to maximize its weekly profit.

4.4 Formulate the Linear Programming Problem. [30%]

4.5 Plot the Constraints and identify the feasible set on that plot. [10%]

4.6 By observing the plot (and doing some simple math) find the Optimal Solution and report the number of each product to produce per week and the corresponding profit. [30%]

These formulas are provided without further context.

$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ...$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1}}$

$$\begin{pmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{pmatrix}$$

$A^T A c = A^T y$

$a_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$

$b_i = y_i$

$$\begin{bmatrix} \frac{2h_0}{3} & \frac{h_0}{3} & 0 & \cdots & 0 & 0 & 0 \\ \frac{h_0}{3} & \frac{2}{3}(h_1 + h_0) & \frac{h_1}{3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{h_{n-2}}{3} & \frac{2}{3}(h_{n-2} + h_{n-1}) & \frac{h_n}{3} \\ 0 & 0 & 0 & \cdots & 0 & \frac{h_{n-1}}{3} & \frac{2h_{n-1}}{3} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} \frac{f_0}{h_0} - \alpha \\ \frac{f_1}{h_1} - \frac{f_0}{h_0} \\ \vdots \\ \frac{f_{n-1}}{h_{n-1}} - \frac{f_{n-2}}{h_{n-2}} \\ \beta - \frac{f_{n-1}}{h_{n-1}} \end{bmatrix}$$

$hf(\frac{h}{2})$

$h[f(0) + f(h)]$

$\frac{h}{3}[f(0) + 4f(h) + f(2h)]$

$x_{n+1} = x(t_n) + \Delta t f(t_n, x_n)$

$x_{n+1} = x_n + \frac{1}{2}\Delta t[f(t_n, x_n) + f(t_n + \Delta t, x_{n+1}^p)]$

$k_1 = \Delta t f(t_n, x_n)$

$k_2 = \Delta t f(t_n + \alpha \Delta t, x_n + \beta k_1)$

$x_{n+1} = x_n + ak_1 + bk_2$

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