

NOTE: Answer **ALL** questions.

Time allowed is **90** minutes.

The exam paper has **3** sections on **9** pages, with a full score of 100 marks.

You are required to use the provided **Examination Book** only for answers.

Section A: Multiple-Choice Questions

This section is worth a total of **50** marks, with each question worth **5** marks.

1. Given a function $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$, then $x = 0$ is a B with respect to $f(x)$.

(A) point of continuity

(B) point of removable discontinuity

(C) point of jump discontinuity

(D) point of infinite discontinuity

Answer: B, the reason is $\lim_{x \rightarrow 0} f(x) = 0 \neq f(0)$

2. When $x \rightarrow \infty$, which of the following is an infinity? D.
- (A) $x \sin x$ (B) $x \cos x$ (C) $x \sin \frac{1}{x}$ (D) $x \cos \frac{1}{x}$

Answer: D For $\lim_{x \rightarrow \infty} \cos \frac{1}{x} = 1$

3. Consider a curve $C: y = \frac{x^2 - x}{|x|(x+1)}$. How many horizontal and vertical asymptotes in total does C have? B.
- (A) 1 (B) 2 (C) 3 (D) 4

Answer: C $y = 1$, $y = -1$ are horizontal asymptotes, $x = -1$ is a vertical asymptote.

4. If $\ln(1 + \alpha x^2)$ and $1 - \cos x$ are equivalent infinitesimals when $x \rightarrow 0$, then $\alpha =$ C.
- (A) -1 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 1

Answer: B

$$\frac{\ln(1+\alpha x^2)}{x^2} = 1 \quad \text{let } t = \alpha x^2$$

$$\ln(1+\alpha x^2) = \alpha x^2 = \frac{1}{2} x^2$$

5. Given

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{(x - a)^2} = 5,$$

which statements in the following is wrong? C(A) $f(x)$ is continuous at $x = a$.(B) $f(x)$ is derivable at $x = a$.(C) $f(a)$ is a local minimum value.(D) $f(a)$ may not an extreme value.Answer: D6. Let $P(x_0, y_0)$ be a point on the curve

$$C: y = x^3 + 3x^2 - 4, \quad y' = 3x^2 + 6x = 3$$

$$y - 3 = -3(x + 1) \Rightarrow y = -3x - 3$$

$$y = -1 + 3 = 2 \Rightarrow y = x + 2$$

where the tangent line at P is perpendicular to a line $L: x - 3y + 2 = 0$.The point $P(x_0, y_0)$ is C.(A) $P(-1, 3)$ (B) $P(1, 2)$ (C) $P(-1, -2)$ (D) $P(1, -3)$ Answer: C7. For a function $f(x) = \sin^4 x$, try to calculate the n -th order derivative $f^{(n)}(x) =$ B.

*Hint:

$$f(x) = (\sin^2 x)^2 = \left(\frac{1 - \cos 2x}{2} \right)^2 = \frac{1 - 2\cos 2x + \cos^2 2x}{4}$$

$$\frac{d^n}{dx^n} [\cos(ax + b)] = a^n \cos(ax + b + n\frac{\pi}{2});$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

$$(A) 2^{n-1} \cos(2x + \frac{n\pi}{2}) + 2^{2n-3} \cos(4x + \frac{n\pi}{2})$$

$$(B) -2^{n-1} \cos(2x + \frac{n\pi}{2}) + 2^{2n-3} \cos(4x + \frac{n\pi}{2})$$

$$(C) 2^{n-1} \cos(2x + \frac{n\pi}{2}) - 2^{2n-3} \cos(4x + \frac{n\pi}{2})$$

$$(D) 2^{n-1} \cos(2x + \frac{n\pi}{2}) + 4^{2n-3} \cos(4x + \frac{n\pi}{2})$$

Answer: B

$$\frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{1}{\sqrt{x^2 + 1}} \cdot 2x\right) = \frac{1}{x + \sqrt{x^2 + 1}} \cdot \frac{x + \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$

Advanced Mathematics (Module 1)

8. Given $f(x) = \ln(x + \sqrt{x^2 + 1})$, there are $dy = \underline{C}$ and $\frac{d^2y}{dx^2} = \underline{\quad}$.

(A) $\frac{1}{\sqrt{x^2 + 1}}$, $\frac{1}{\sqrt{x^2 + 1}^3}$;

(B) $\frac{1}{\sqrt{x^2 + 1}}$, $-\frac{1}{\sqrt{x^2 + 1}^3}$;

(C) $\frac{1}{\sqrt{x^2 + 1}}dx$, $-\frac{x}{\sqrt{x^2 + 1}^3}$;

(D) $\frac{1}{\sqrt{x^2 + 1}}dx$, $\frac{x}{\sqrt{x^2 + 1}^3}$.

Answer: C

$$\frac{d}{dx} \frac{1}{\sqrt{x^2 + 1}} = -\frac{1}{2} \cdot \frac{1}{\sqrt{x^2 + 1}^3} \cdot 2x = -\frac{x}{\sqrt{x^2 + 1}^3}$$

$$= -\frac{1}{2} \cdot \frac{1}{\sqrt{x^2 + 1}^3} \cdot 2x$$

9. Which of the following statements is wrong? B

(A) If $f(x)$ is continuous on a closed interval $[a, b]$, and $f(a)f(b) < 0$, then there exists at least one $\xi \in (a, b)$ such that $f(\xi) = 0$.

(B) If $f(x)$ is continuous on an open interval (a, b) , then $f(x)$ is derivable on the open interval (a, b) .

(C) If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ have both a global maximum and a global minimum value.

(D) $f(x)$ is derivable on an open interval (a, b) , and $x_0 \in (a, b)$ is a point of extreme value, namely, $f(x_0)$ is a local maximum or minimum value with $f'(x_0) = 0$.

Answer: B

$$f'(x) = \frac{a}{x} + 2bx - 3$$

$$f'(1) = a + 2b - 3 = 0$$

$$f'(2) = \frac{a}{2} + 4b - 3 = 0$$

$$\begin{cases} 2a + 4b - 3 = 0 \\ \frac{a}{2} + 4b - 3 = 0 \end{cases}$$

$$\frac{2}{2}a - 3 = 0 \quad a = 2$$

$$4b = 2 \quad b = \frac{1}{2}$$

10. Suppose $f(x) = a \ln x + bx^2 - 3x$ has extreme values at $x = 1$ and $x = 2$. Then the values of a, b are B

(A) $a = \frac{1}{2}$, $b = 2$

(B) $a = 2$, $b = \frac{1}{2}$

(C) $a = -\frac{1}{2}$, $b = 2$

(D) $a = -2$, $b = -\frac{1}{2}$

Answer: B

$$\frac{(1+n)^n}{2(n^2+1)} \cdot \frac{\frac{1}{n} + 1}{2(1 + \frac{1}{n})} = \frac{1}{2}$$

Section B: Gap-Filling Questions

This section is worth a total of **25** marks, with each question worth **5** marks.

11. Find the limit

$$\lim_{n \rightarrow \infty} \frac{1 + 2 + \dots + n}{n^2 + n} = \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)}{2}}{n^2 + n} = \lim_{n \rightarrow \infty} \frac{\frac{n+1}{2}}{\frac{1}{n} + 1} = \frac{1}{2}$$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n^2 + 1} + \frac{2}{n^2 + 2} + \frac{3}{n^2 + 3} + \dots + \frac{n}{n^2 + n} \right) = \frac{1}{2}.$$

*Hint: try to use the sandwich theorem.

Solution.

$$\frac{1 + 2 + \dots + n}{n^2 + n} \leq \frac{1}{n^2 + 1} + \frac{2}{n^2 + 2} + \frac{3}{n^2 + 3} + \dots + \frac{n}{n^2 + n} \leq \frac{1 + 2 + \dots + n}{n^2 + 1}$$

and $\lim_{n \rightarrow \infty} \frac{1 + 2 + \dots + n}{n^2 + n} = \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)}{2}}{n^2 + n} = \frac{1}{2}, \lim_{n \rightarrow \infty} \frac{1 + 2 + \dots + n}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)}{2}}{n^2 + 1} = \frac{1}{2},$

By Sandwich Theorem,

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n^2 + 1} + \frac{2}{n^2 + 2} + \frac{3}{n^2 + 3} + \dots + \frac{n}{n^2 + n} \right) = \frac{1}{2}.$$

$$\frac{x^2 - ax - ax^2 + b + bx}{1 + x} = 0.$$

$$\lim_{x \rightarrow \infty} \left(\frac{x^2}{1 + x} - ax + b \right) = 0, \quad \frac{(b-1)x + b}{1 + x} = 0.$$

then a and b are evaluated as $a = \underline{\hspace{2cm}}, b = \underline{\hspace{2cm}}.$

Solution.

$$\lim_{x \rightarrow \infty} \left(\frac{x^2}{1 + x} - ax + b \right) = 0 \text{ i.e. } \lim_{x \rightarrow \infty} \left(\frac{(1-a)x^2 + (b-a)x + b}{1 + x} \right) = 0,$$

So, the coefficients of x^2 and x are both zero. $a = 1, b = 1$

Advanced Mathematics (Module 1)

13. If $f(x)$ is derivable at $x = 0$ and $f'(0) = 2$, then

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{\sqrt[3]{1+x} - 1} = \underline{6}$$

$$\frac{\frac{1}{3}x}{(1+x)^{\frac{1}{3}} - 1} = \frac{1}{3}$$

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{\frac{1}{3}x} = 3f'(0)$$

Solution.

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{\sqrt[3]{1+x} - 1} = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{\frac{x}{3}} = 3f'(0) = 6$$

14. For a function

$$y = x \cos 2x,$$

compute the higher order derivative $y^{(2023)} = \underline{\hspace{2cm}}$.

Solution.

$$\begin{aligned} (x \cos 2x)^{2023} &= x \cdot (\cos 2x)^{2023} + C_{2023}^1 x' \cdot (\cos 2x)^{2022} \\ &= x \cdot 2^{2023} \cos(2x + 2023 \cdot \frac{\pi}{2}) + 2023 \cdot 2^{2022} \cos(2x + 2022 \cdot \frac{\pi}{2}) + C_{2023}^2 (\cos 2x)^{2021} \\ &= x \cdot 2^{2023} \sin 2x - 2023 \cdot 2^{2022} \cos 2x \end{aligned}$$

$$\begin{aligned} y' &= \cos 2x + x \sin 2x \cdot 2 \\ &= \binom{2023}{0} x^{(0)} (\cos 2x)^{(2023)} + \binom{2023}{1} x^{(1)} (\cos 2x)^{(2022)} \\ &= \binom{2023}{0} x (\cos 2x)^{(2023)} + \binom{2023}{1} x' (\cos 2x)^{(2022)} \\ &= x \cdot 2^{2023} \cos(2x + 2023 \cdot \frac{\pi}{2}) + 2023 \cdot 2^{2022} \cos(2x + 2022 \cdot \frac{\pi}{2}) + C_{2023}^2 (\cos 2x)^{2021} \\ &= x \cdot 2^{2023} \sin 2x - 2023 \cdot 2^{2022} \cos 2x \end{aligned}$$

15. Given

$$\begin{cases} x = te^t, \\ y = e^{2t}, \end{cases}$$

try to compute

Solution.

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} \\ \frac{dy}{dt} &= 2e^{2t}, \quad \frac{dx}{dt} = e^t + t \cdot e^t = e^t(1+t) \\ \frac{dy}{dx} &= \frac{2e^{2t}}{e^t(1+t)} = \frac{2e^t}{1+t} \\ \frac{d^2y}{dx^2} &= \frac{\left(\frac{2e^t}{1+t} \right)'}{e^t(1+t)} = \frac{2te^t}{(1+t)^3} = \frac{2te}{(1+t)^2} \end{aligned}$$

$$\frac{d^2y}{dx^2} = \frac{d \left(\frac{dy}{dx} \right)}{dx} = \frac{d \left(\frac{dy}{dx} \right)}{dt} \cdot \frac{dt}{dx}$$

Page 5 of 9

$\frac{d^2y}{dx^2}$ 不是 $\frac{dy}{dx}$ 的二阶导吗? 为什么又除 $(1+t)e^t$

Advanced Mathematics (Module 1)

Section C: Extended Answer Questions

This section is worth a total of **25** marks, with the marks of every question as shown.

16. (7 marks) For a function

$$y = \left(\sqrt{\frac{x}{x+1}} \right) \cdot x^{\sin x}, \quad \frac{1}{y} \cdot y' = \frac{1}{2x} - \frac{1}{2(x+1)} + \cos x \ln x + \sin x \cdot \frac{1}{x}$$

try to use the *logarithmic differentiation method* to compute $\frac{dy}{dx}$.

Solution.

Taking logarithm on both sides, we get

$$\ln y = \frac{1}{2} [\ln x - \ln(x+1)] + \sin x \cdot \ln x$$

Differentiate with respect to x on both sides, we have,

$$\frac{y'}{y} = \frac{1}{2} \left[\frac{1}{x} - \frac{1}{x+1} \right] + \cos x \ln x + \frac{\sin x}{x}$$

So,

$$\frac{dy}{dx} = \left(\sqrt{\frac{x}{x+1}} \right) \cdot x^{\sin x} \cdot \left\{ \frac{1}{2} \left[\frac{1}{x} - \frac{1}{x+1} \right] + \cos x \ln x + \frac{\sin x}{x} \right\}$$

Advanced Mathematics (Module 1)

17. (5 marks) Compute the following limit:

$$\lim_{x \rightarrow 0} \frac{e^x - x - 1}{x \sin x} = \frac{e^x - x - 1}{x^2} = \frac{e^x - 1}{2x} = \frac{1}{2}$$

Solution.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^x - x - 1}{x \sin x} &= \lim_{x \rightarrow 0} \frac{e^x - x - 1}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{e^x - 1}{2x} \\ &= \lim_{x \rightarrow 0} \frac{e^x}{2} \\ &= \frac{1}{2} \end{aligned}$$

Note: For Questions 18 & 19 below, select **ONLY ONE** to answer.

18. (5 marks) Given a sequence $\{a_n\}$, with $a_1 = 1$ and $a_{n+1} = \sqrt{a_n + 12}$, show

- (a) $\lim_{n \rightarrow \infty} a_n$ exists;
 (b) Find the value of the limit $\lim_{n \rightarrow \infty} a_n$.

Solution. $a_1 = 1 < 4$, $a_2 = \sqrt{a_1 + 12} = \sqrt{13} < 4$, and $a_1 < a_2$, so $a_2 = \sqrt{a_1 + 12} < \sqrt{a_2 + 12} = a_3$, inductively, $a_n < a_{n+1}$, and if $a_n < 4$, then $a_{n+1} = \sqrt{a_n + 12} < \sqrt{4 + 12} = 4$. So the sequence $\{a_n\}$ is monotonically increasing and bounded above. By MST, we know that $\lim_{n \rightarrow \infty} a_n$ exist. Say $\lim_{n \rightarrow \infty} a_n = A$, taking limits on both sides with respect to $a_{n+1}^2 = a_n + 12$, we get $A^2 = A + 12$. Solve the equation, we get $A = 4$ or $A = -3$. As $a_n > 0$, so $\lim_{n \rightarrow \infty} a_n \geq 0$. $A = 4$

Advanced Mathematics (Module 1)

19. (5 marks) Use the concavity property to prove

$$8 \frac{(x^4 + y^4)}{2} > \frac{(x+y)^4}{2}$$

Solution.

Let $f(t) = t^4$, and $f'(t) = 4t^3$, $f'' = 12t^2 \geq 0$, so the curve $f(t)$ is concave up, i.e.

$$f\left(\frac{x+y}{2}\right) < \frac{f(x) + f(y)}{2}$$

That is

$$\left(\frac{x+y}{2}\right)^4 < \frac{x^4 + y^4}{2}$$

Making transformation, we have

$$8(x^4 + y^4) > (x+y)^4$$

support $f(t) = t^4$
 $f'(t) = 4t^3$

$$\forall x \neq y.$$

$$f'(t) = 12t^2$$

$$\therefore f''(t) > 0 \therefore f(t) \text{ is Concave Upward}$$

$$\therefore \frac{x^4 + y^4}{2} > \frac{(x+y)^4}{2}$$

$$\therefore 8(x^4 + y^4) > (x+y)^4$$

20. (8 marks) Suppose $f(x)$ has a continuous second order derivative over the interval $[0, 1]$, and $f(0) = f(1) = 2$. Show that there exists at least one $\xi \in (0, 1)$ such that

$$\therefore f(0) = f(1) = 2 \quad 2f'(\xi) + \xi f(\xi) = 0.$$

$$\therefore \exists \eta_0 \in (0, 1) \quad f'(\eta_0) = 0$$

Solution.

Because $f(0) = f(1) = 2$, by *Rolle'* Theorem, there exists a $\delta \in (0, 1)$ such that $f'(\delta) = 0$. Let $F(x) = x^2 f'(x)$, $F(\delta) = 0 = F(0)$, $F(x)$ satisfies three conditions on $[0, \delta]$ of *Rolle'* Theorem, therefore there is a $\xi \in (0, \delta) \subset (0, 1)$ such that $F'(\xi) = 0$. That is,

$$2\xi f'(\xi) + \xi^2 f''(\xi) = 0$$

, as $\xi \neq 0$, cancel the common factor ξ , we have

$$2f'(\xi) + \xi f''(\xi) = 0$$

Advanced Mathematics (Module 1)

USEFUL FORMULAE

$$\frac{d}{dx}C = 0$$

$$\frac{d}{dx}x^\mu = \mu x^{\mu-1}$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2}$$